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The accurate construction of the right angles of the Great Pyramid's ground plan

Louis Clerc

Abstract:

Ancient Egyptian surveyors constructed 90-degree angles at the corners of the Great Pyramid to an accuracy of one part in ten-thousand. This paper proposes that the surveyors achieved this reliably by using an approximation technique and measuring rods and extending the resulting perpendicular lines along the pyramid's sides. Computations based on realistic and testable assumptions yield results that are persuasively close to those observed archaeologically. Using a 20 by 30 m base/side isosceles survey triangle to construct the perpendiculars at the right-angled corners produces a resultant angular deviation of 35.6 arc seconds, compared to the measured average of 37 arc seconds. Similarly, the calculated difference in the length of the sides is 3.93 cm compared to the measured differences in the lengths between the northern and western sides of 4.4 cm and between the northern and the eastern sides of 4.1 cm. Further discoveries at the pyramid's base dating to the appropriate era and found in the appropriate locations also support the historical use of the method. Additional considerations show how sophisticated geometrical intuition was developed during the 4th dynasty and that it was fundamental to the construction of highly symmetrical pyramids.

Introduction

Although Khufu's pyramid was aligned to the cardinal points with an accuracy better than 4 arc minutes,[1] ancient surveyors constructed the 90-degree corner angles with even greater accuracy. Surveyed data now indicates that this was achieved to one-ten-thousandth of 90 degrees, or under 1 arc minute.[2] Until now, there has been no satisfactory answer to how they constructed right angles with such unbelievably low uncertainty. To date, no calculations have been carried out that confirm the viability of any suggested techniques through a correlation of predicted values and observed archaeologically measured values. On the contrary, some pyramid specialists either doubt that the required precision referred to was indeed achieved[3] or simply explain it away as "they obviously mastered the practice of exact survey."[4] That the measured deviation of the corner angles is 5 times smaller than the deviation of the pyramid's side alignments from the average length indicates that a particular angular measuring technique was most likely used. To investigate this, the historical context within which the 4th dynasty construction took place and the geometrical fundamentals involved in the process were studied. The conclusions of these studies are presented here for the first time. The paper presents and develops a new concept that avoids all the objections raised

1 Dash (2017), p. 1; Lightbody (2020a), p. 31.
2 0°37" / 90° = 1,1 10^{-4}. Cf. footnote 28.
3 Stadelmann (1991), p. 220.
4 Müller-Römer (2011), p. 377. Translated from German by the author.

regarding previous proposals, such as the low dimensional and directional stability of the available tools, the absence of a clearly-stated geometric criterion as a final objective, dependency on other right-angle constructions, or overly-cumbersome procedures. This paper utilizes a trigonometric model and the law of error propagation to evaluate if a two-step approximation technique employing measuring rods (ATMR) and the subsequent extension of the perpendiculars to the opposite side (EP) of the pyramid yields the expected result. The following wide-ranging discussion outlines how useful such a method would have been for the ancient Egyptian people in the absence of advanced theoretical knowledge. The analysis concludes that the use of the technique has a sound historical basis. It also investigates how the use of the method may relate to the later use of similar methods in the context of the Greek culture.

Detecting the pyramid's ground plan

Originally, the Great Pyramid of Giza was completely covered in smoothed Tura limestone casing blocks. Only a few of the casing stones are still extant along the ground level platform and these are weather-worn and often badly damaged. Almost all are located in the middle sections of the sides. The original dimensions of the casing can no longer be established with absolute certainty, as the corners are gone today, but their positions can be reconstructed by intersecting the extrapolated reconstructed baselines out to the corners. For the sake of reliability, only survey work carried out after all four sides were cleared of rubble and other debris has been considered in this paper. Several points along the original edges can be found and fitted lines drawn through them. The difference in the azimuths between the two crossing baselines at each corner determines the corner angle.[5] To obtain the best data set, Lehner identified 84 points along traces of the original baselines left by the edges of the casing's footprint on the platform stones, which he discovered during a meticulous search.[6] He mapped them on the grid of the Giza Plateau Measuring Project (GPMP).[7] In 2015, Dash repeated the survey and also mapped them onto the control grid of the GPMP. He further defined the baselines as being the best fit line passing through these points using the statistical method of linear regression analysis. By extrapolating and crossing these lines, he determined the corner positions within an area of a few centimeters.[8] He subsequently calculated the distances between the neighboring corners as well as the lengths of the diagonals, and most importantly for this study, the angles at the crossings of the sides and diagonals (fig. 1-a.). A similar survey had already been completed by Dorner, and of the eight angular deviation value ranges calculated for the corner right angles measured by Dash and Dorner, seven are under 60 arc seconds, and the overall average is 45 arc seconds. The values of the side lengths and relevant angles are compiled and set out in fig. 1.

5 Dorner (1981), p. 75.
6 Dash (2012), p. 13.
7 Lehner (2020).
8 Dash (2015), p. 10.

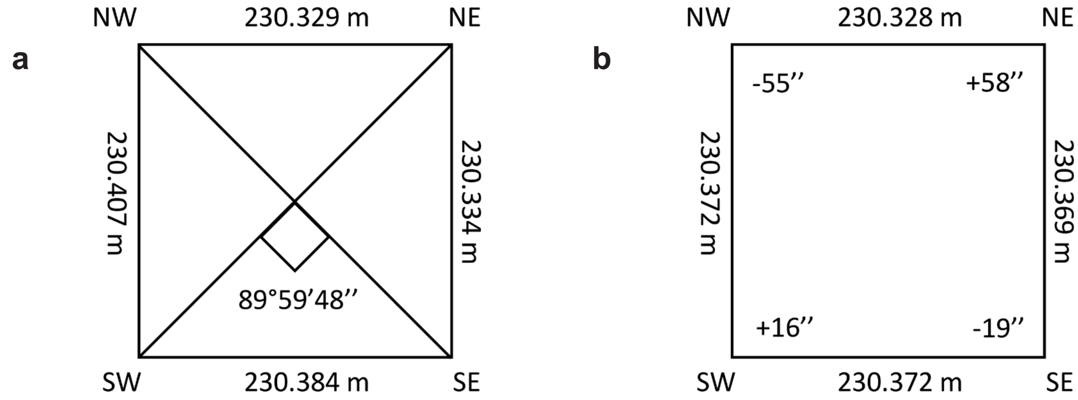

Fig. 1.
a. Ground plan square defined by the casing baselines according to Dash. The diagonals connect the opposite corners of the casing (redrawn and modified from Dash).
b. Ground plan square with sides as measured by Dorner; deviations of the corner angles from 90 degrees calculated from the sides' azimuths. A perfect square is defined by 4 equal sides and 4 right angled corners leading to a perfect 90-degree angle of the crossing diagonals. Thus, the accuracy of the crossing angle is a substitute representing the accuracy of the corner angles. The difference between the average side lengths surveyed and calculated by Dash and Dorner was only 3 mm over 230.36 m, and this is another indicator of the accuracy of the base plan square. These values justify the conclusion that the corner angles of the casing were repeatedly and reliably achieved with an uncertainty of less than 60 arc seconds.

Existing theories and their shortcomings

The survey data indicates that the eastern or western side[9] of the pyramid was first aligned towards the north with a deviation of 2 arc minutes and 47 arc seconds[10] and that its length was probably determined using measuring rods. The orientation of the side to the cardinal points will not be discussed further here because the process of constructing of the 90-degree angles to set out the subsequent sides based on this first side is independent of the procedure used to orient the first side to the cardinal points. Several practical techniques have been proposed that the ancient surveyors could have used to measure out the right angles. Though theoretically equivalent in intent, the methods differ significantly in practice and in their ability to achieve a given minimum uncertainty. A specific objection can be put forth arguing against the use of each method, as follows. Unterberger proposed construction using an auxiliary rectangle with diagonals positioned adjacent to the pyramid's ground level edge (fig. 2-a).[11] The challenge there is to set out the long diagonals accurately. To do this, the midpoint of the rectangle Pm must first be fixed with as little uncertainty as possible. This can be done either by drawing intersecting arcs with centers at the endpoints of the side, or by constructing a bisecting perpendicular extending out from the pyramid's base edge, along which the position of the center point PM can be freely chosen. The half-diagonals running from the corners to the point Pm are then completed to form diagonals defining the auxiliary rectangle and with it, the 90-degree angles at the pyramid's corners. This would be a laborious process

9 Spence (2000), p. 321.
10 Dorner (1981), p. 77.
11 Unterberger (2008), p. 81.

demanding an additional construction step for creating the right angle at the center of the base, or to locate the point Pm, as well as extended diagonal measurements (over 230 m). The method is too complicated and prone to introducing further inaccuracies. Another procedure is to use a triangle with sides in the ratio 3-4-5 (fig. 2-b). This method certainly was and is useful for building every-day structures such as houses, however, Dorner's analysis of the minimum number of steps required to obtain the archaeologically observed right angles[12] using this method showed that the necessary measuring precision required when using this method was twice as high as required when using a method that used intersecting arcs (fig. 2-c). The intersecting arc technique is impressive in its simplicity and it can be applied directly at the pyramid's corners, but unfortunately, intersecting arcs must be drawn with cords, the elasticity of which make the use of such a method unlikely. The fourth possible method employs a wooden building square that is flipped by 90°, as proposed by Engelbach.[13] When compared to fig. 2-c, it is clear its use in this way produces an upside-down and extremely slender version of the isosceles triangle. In the current author's opinion, Engelbach's idea was ingenious in that it started with a dimensionally stable tool and eliminated the wooden square's inherent manufacturing error via the method of application. However, in a practical trial Engelbach was not able to achieve results of better than 1.5 arc minutes. Moreover, the perpendicular was not yet extended over 230 m to the opposite corner. The experimental application of the method, therefore, led to results with errors much larger than the archeologically observed tolerance of 1 arc minute (fig. 2-d).

Both wooden rods and cords made of natural fibers were available to measure distances. The latter had a lower resistance to external influences such as humidity change, temperature change, and strain, especially longitudinally in the measuring direction. According to Shazad, cords yield longitudinally up to 2% against the strain, equating to 104 cm over 100 cubits (52.4 m).[14] In a practical trial, Unterberger noted a lengthening of 1 m over 60 m and also noted the very arduous handling process.[15] To draw the arcs of a circle, the rope must be held tightened above the ground to avoid sagging and to evade obstacles. For precision, it must be absolutely taut. Without a high traction force this is impossible. As the ancient Egyptians could not accurately determine and maintain this type of force, they could not have used it consistently. This means that identical measurements were not repeatable. The knotting of the cord at regular intervals would have only worsened the situation. In conclusion, the measuring cord could not achieve a deviation of only a few centimeters over 230 m. Even so, it is not useless with respect to its directional stability. When held stretched it does not veer sideways and allows the marking of an accurate directional alignment. This means that leaving the measurement of the length to other, more precise longitudinal methods make good sense.

Rods are also subject to changes in temperature and air moisture content levels but they are more stable longitudinally. It has been noted that if they are cut across the wood grain they are ten times more stable longitudinally than rods cut along the grain in the measuring direction. When several such rods are connected together, long distances can be measured very accurately (fig. 5).

Despite the many objections, all the existing theories do meet the most elementary precondition. They are all able to apply one of the three possible fundamental geometrical construction methods yielding a right angle at the corners, namely, Euclid's definition, the Pythagorean theorem, or Thales' theorem (fig. 2). In practice, however, none of these theoretical methods can compensate for the negative influences encountered in the real physical world on the measuring process, and

12 Dorner (1981), p. 109, p. 113.
13 Clarke, Engelbach (1999), p. 67.
14 Shazad (2013), fig. 11.
15 Unterberger (2008), p. 17.

none of them allow for the accurate extension of the perpendicular along the side. The current proposals all focus on the construction of a perpendicular on the pyramid's side that is far shorter (estimated at 30 m for practical reasons) than the pyramid's edge length. The constructed perpendicular must still then be extended over 230.36 m. This task cannot be taken for granted. It must also be surveyed and further errors will then arise. In most cases the small error of 60 arc seconds or less observed in the archaeology of the site is already accounted for by the deviation due to the construction of the 90-degree angles at the corners, leaving no room for error during the linear extension along the sides. A comprehensive workable theory must consider both parts of the task. Even so, there are intelligent concepts behind each existing theory that stimulated the development of the novel method proposed in this paper. The essence of the new theory is that once a suitably accurate linear measurement protocol had been devised, the bisecting perpendicular method shown in fig. 2-c[16]. enabled the ancient surveyors to execute the procedure with the archaeologically observed accuracy.

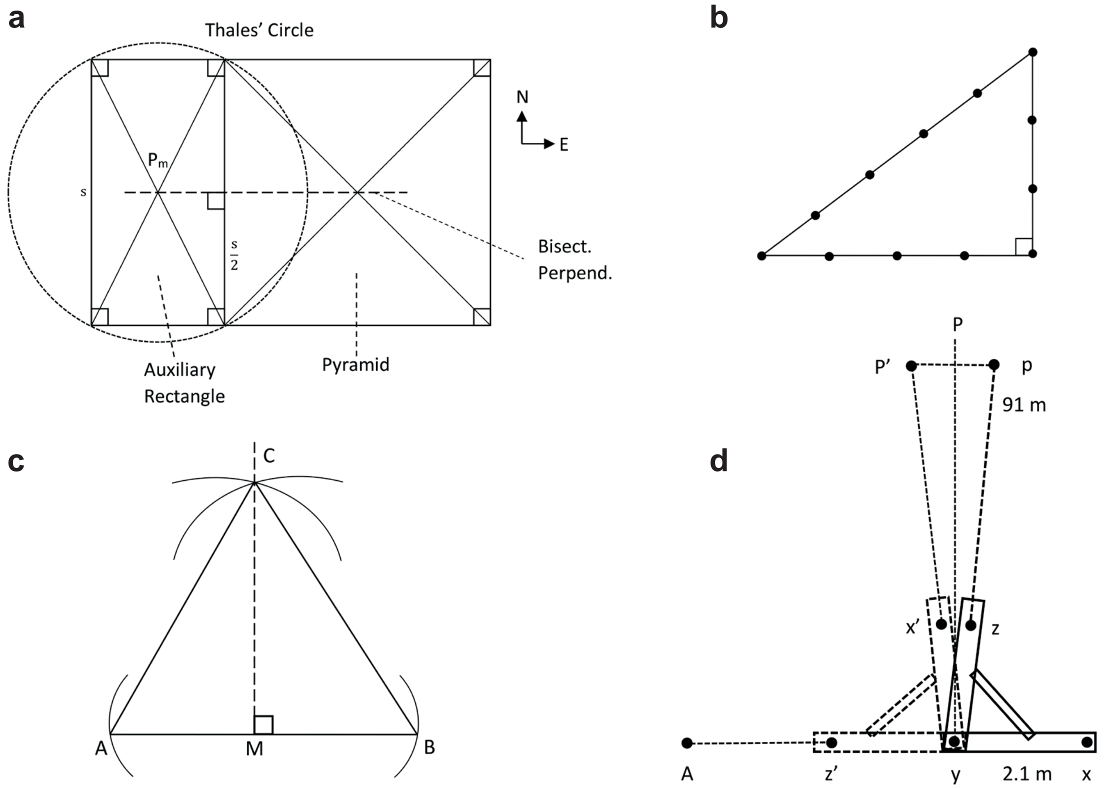

Fig. 2. Practical construction techniques for 90-degree angles.
a. A circle is drawn through the diagonals of an auxiliary rectangle also illustrating the application of Thales' theorem (redrawn and modified from Unterberger).
b. The 'Egyptian triangle' constructed with measuring cords (Pythagorean theorem and triplet).
c. Unilateral intersecting arcs with measuring cords (based on Euclid's definition). $\overline{AM} = \overline{BM}$; $\overline{AC} = \overline{BC}$.
d. A flipped building square. Engelbach described the one he used in the field. The length of the legs was about 7 feet (2.1 m). It was not clear how the segments defined by the square legs of 2.1 m were extended to 91 m. In sum, the method cannot replicate the archeologically observed precision of less than 1 arc minute deviation over 230 m (redrawn from Clarke and Engelbach 1999, p. 67).

16 Robins, Shute (1987), p. 47.

The approximation technique with measuring rods

The ATMR proposed here is based on Euclid's definition of a right angle, and proposition 11 of Book 1 of Elements.[17] After aligning the first side of the pyramid and fixing its endpoints, the straight line is extended laterally beyond the pyramid's corner M (the horizontal line on fig. 3). Points A and B are then marked at the same distance on either side of the pyramid's corner M. Point C_1 is then placed at an appropriate distance from the corner. After measuring the distances between C_1 and A as well as C_1 and B, point C is shifted toward the longer leg's side. This measurement process is repeated until equidistance is reached at C_d. The connection between C_d and the corner M is thus a perpendicular line whose extension at a right angle to the first side yields the second side, at the end of which the next corner can be fixed. The method is applied again and again until the pyramid's entire perimeter ground plan is established. All linear measurements, from C_1, C_2, and C_d, to A and B, as well as from M to A and B, are performed exclusively using longitudinally stable wooden measuring rods (fig. 3). These are placed alongside stretched-out directionally stable cords (fig. 5). All of the equivalent equal distances must be measured out following the same protocol (equal numbers and identical application sequence of the same individual rods, preferably only the same two). This procedural consistency improves the end results.

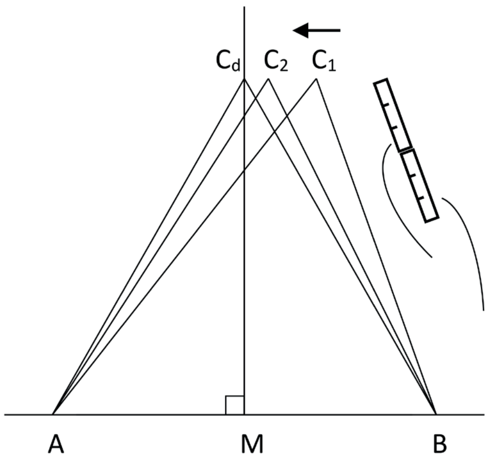

Fig. 3. The approximation technique with measuring rods (ATMR). A shift parallel to the base is not required.

Documentary evidence of such knowledge

The use of the method described above could indicate that the ancient Egyptians were already aware of a principle only described much later by Euclid's Elements of Geometry in proposition 11 of Book 1 of his work Elements. That was written around 300 BC, many centuries after the construction of Khufu's pyramid. No surviving document from the Old Kingdom mentions the use of such methods, however, the Rhind Mathematical Papyrus (RMP – pBM 10057&10058) dating from approximately 1,550 BC does include complex geometric algorithms. According to the scribe Ahmes, he had copied it from an earlier version dating from 1,850 BC, during the reign of the pharaoh Amenemhet of the 12th dynasty, only 700 years after the time of Khufu.

17 Euclid, (2007), Elements, Book 1, pp. 6, 16.

RMP problems 56-59 are concerned with pyramid measurement procedures. The examples demonstrate how to calculate the slopes of the pyramid's sides. The value called the seqed (or seked) of the slope was the reciprocal of the slope and was given by the run in relation to a 1-cubit rise. The illustrating figures of pyramids lack any lines referring to the height of the monuments, but the accompanying text reads "the seqed is taken to be half the width of the base divided by the height..."[18] Furthermore, in problems 57-59, the seqed quoted is the same as that of Khafre's pyramid.[19] In these examples the height was undoubtedly used as the bisecting perpendicular of an isosceles triangle that formed the vertical cross section of the pyramid. When considered together with the dimensions in problem 56 that correspond to those of a large Old Kingdom pyramid, it seems most likely that this document reflects knowledge that came down from the Old Kingdom.

The close relationship between the pharaohs, state rituals, and the state's pharaonic architecture is also documented on the Palermo Stone. Several entries reference the building of temples and the laying out of ground plans using the "stretching of the cord" ritual.[20] A Pyramid Text also includes reference to the 'establishment and encircling' of a pyramid, an action that resembles the idea of an encircling survey to establish the ground plan of a monument at the start of construction."[21]

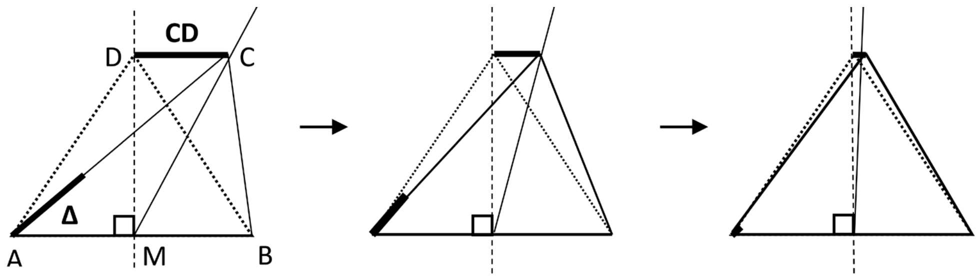

Fig. 4. Construction of the bisecting perpendicular through M using the surveying triangle: depiction of the correlation between the angular deviation at the apex and the difference in the lengths of the legs during the procedure. Δ is the leg length difference, CD is the lateral deviation from the ideal perpendicular, and M is the pyramid's corner.

Validation of approximation technique with measuring rods and extension of perpendicular

The reason that the ATMR procedure together with the EP (extension of the perpendicular described below) can successfully replicate the accuracy observed in the monument's ground plan is the result of an elementary geometrical relationship. For a scalene triangle, where all the three sides are of different length, but of similar magnitude, the difference in length between the two legs (Δ) and the lateral deviation CD of the apex from the perpendicular bisecting its base, are close to equal. In the special case where the base length will also be equal to the final leg lengths (equilateral), this is in fact described by the equation $\Delta = CD$ (see equation 5b[22] in the appendix

18 Robins, Shute (1987), p. 47.
19 Robins, Shute (1987), p. 47.
20 Wilkinson (2000), p. 111-112.
21 Lightbody (2020b), p. 57-59.
22 To illustrate, the equilateral special case of the intended isosceles triangle was chosen because the length difference and apex deviation were equal. This can be seen in more detail in fig. 11. and equations 5b for the equilateral and 5a for the more general case.

along with the more complex mathematical demonstrations). Fig. 4. demonstrates the basic principles that underpin the survey work. The work begins with the selection of a provisional apex point, roughly chosen at an appropriate distance from the corner M to ensure adequate accuracy. When a difference in length between the triangle's two legs is observed, the surveyor knows that the triangle is still scalene. As a result, the apex point is shifted in the appropriate direction, and the comparison of the legs is repeated until they no longer have a difference in length. This is the final objective criterion. The bisecting perpendicular is then constructed by joining the apex to the bisection point at the base, which was the corner point M defined initially. In reality, the resultant form is not an ideal isosceles triangle, but the remaining errors can be evaluated. In fig. 4, the evolution of the form and the trigonometric laws behind the procedure can be seen as the method unfolds during the implementation of the ATMR. From the measured angular deviation observed at the archaeological site, the level of uncertainty left behind by the length measurements used on the construction site can be appreciated. The measuring techniques used can subsequently be guessed at and the viability of any proposed method checked by an appropriate computation.

Carrying out measurements on the construction site

The proposed ATMR technique is relatively straightforward, however, establishing exactly how it was implemented on the construction site poses some remaining challenges. First, the procedure used must have a high degree of directional and dimensional stability. This must have been dealt with by choosing the tools to implement the procedure carefully. Second, the scale of the pyramid means that the measurements could not have been carried out in one operation. The task must have been split up into several successive steps. Third, measurements in the real world are physical procedures, and as such, they are inevitably subject to errors. Therefore, error analysis of some description would have been necessary.

With respect to the selection of tools, it is concluded here that measurement rods were used and that the extended distances must have been partitioned up and every subsection measured separately. In the measurement of each subsection, a small error arises. The exact value of the error is unknown but the order of its magnitude can be estimated through repeated observations (fig. 6). Due to the procedure followed during the implementation of the ATMR, the number and length of subsections in both legs are equal or close to equal. It is reasonable to assume that each error was of the same order of magnitude, although individually different and of varying algebraic sign.[23] Each new error was summed and passed along during the implementation of the procedure up to the end of the measurement. At first glance the more numerous the single measurements were, the harder it would be to achieve a certain accuracy. Yet, in some subsections the random errors are positive and in others, negative, thus, partially canceling each other out. This works in the surveyor's favor and makes the quest for high accuracy more achievable. The law of error propagation is also based on the same principle, and as a result it is possible to estimate the influence of all the errors on the final outcome, starting from the beginning of the procedure and continuing up to the end of the measurements. It is not necessary to fully understand the abstract mathematics involved, but the underlying principle is demonstrated in figs. 5, 6 and 7 below. These describe a measuring technique as it was most likely known and used during the 4th dynasty, based on the availability of the tools involved and the simplicity of handling them.

The maximum level of accuracy or minimum level of uncertainty is determined by two crucial parameters, namely, the precision of the tools used and the law of error propagation. The precision

23 Dorner (2007), p. 50.

is dependent on material properties such as the flexibility of cords and wooden rods, physiological constraints of the operator (including of the human eye), and the measuring protocol used. Finally, the different factors must work together to produce a resultant Δ small enough to match or exceed the archaeologically observed angular deviation.

The accurate measurement of a distance between two points

Ultimately, the construction of right-angled corners with such high levels of accuracy relies on accurate linear measurements. The effectiveness of the rod–cord combination technique is best assessed by evaluating the ratio of the total error compared to the total measured length. This is called relative accuracy (RA). The lower the value, the higher the precision. Clearly, the RA of the constituent steps must be lower than that for the eventual outcome. For the corner angles this was 1×10^{-4} (see footnote 2). In fact, the measurement accuracy achievable with dimensionally stable rods is astonishing. With 8-cubit rods (4.2 m), interface clearances of 0.25 mm, a stable temperature, and relatively dust-free surroundings, for a pyramid's side of 230.36 m, the achievable error is 1.85 mm (RA = 8×10^{-6}).[24] Similarly, for a surveyor's isosceles triangle with sides of 30 m, the error is 0.667 mm (RA = 2.25×10^{-5}). This demonstrates that longer distances can be measured with higher precision and that with smaller orders of magnitude of the RAs, achieving the small angular values for deviations observed at the pyramid's corners is readily achievable.

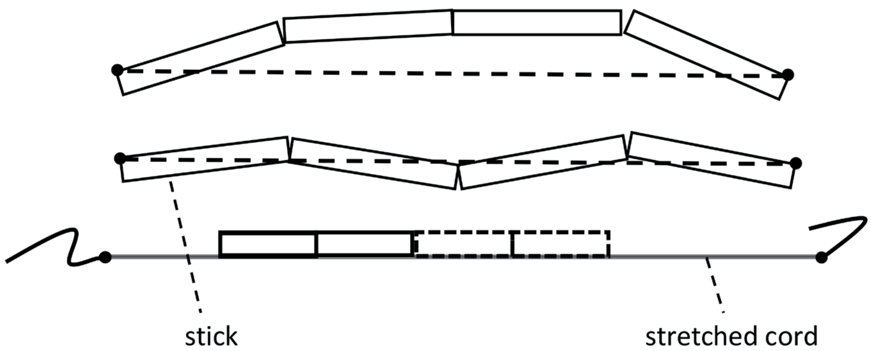

Fig. 5. Directional stability using measurement with rods aligned alongside a stretched rope. The rope is a directional guide that prevents curvature and zigzagging and ensures that the shortest line connecting the ends is measured, and thus, that it is a straight line. The rope is not used as a linear measurement device.

24 440 cubits side length gives 55 interfaces between the ends of rods of 8 cubits. If there is a 0.25mm linear error between each rod, the resultant error propagates to only 1.85 mm = 0.25mm x 55^0.5.

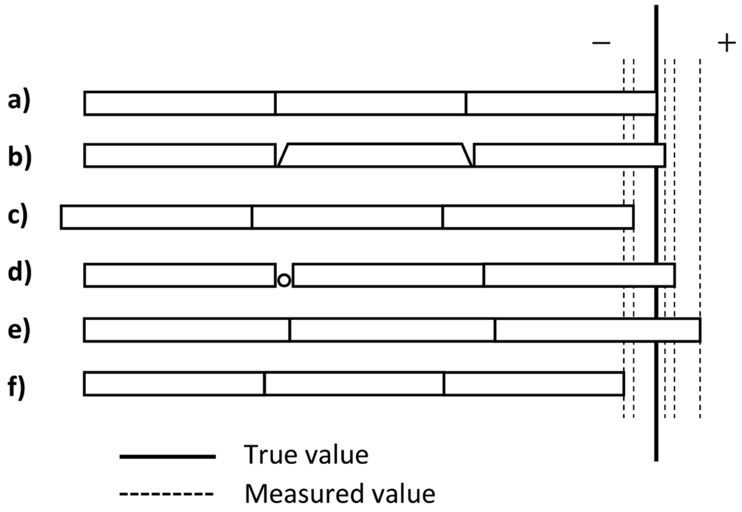

Fig. 6. The types of random errors that can occur when using dimensionally stable rods. Not to scale. The resulting uncertainties are positive in some cases and negative in others, therefore, they partially cancel each other out in cases of successive measurements. a. The intended true value, b. Joint gaps due to incongruent end cuts and unequal pressure applied when joined together, c. Inadvertent backward shifting of the initial position, d. Fine sand grain contamination, e. Thermal expansion and extension, and f. Thermal shrinking and shortening. The total error does not increase in direct proportion to the number "n" of subsections, but with the square root √n. With increasing n, the relative accuracy (RA) becomes more precise. Simply put, with 100 subsections, the distance grows 100 times but the total error only 10 times.

The accurate extension of a straight line in segments over a defined distance

This is achieved by setting out and aligning a series of straight line segments using ranging poles positioned at certain intervals and by sighting along them with a given visual acuity.[25] With each sighting, a small random angular error arises that obeys the law of error propagation. The angular errors are perpendicular to the measuring direction (fig. 7), and with a constant visual acuity they are directly proportional to the distance between the ranging poles. This leads to a surprising consequence. The total angular accuracy can be enhanced by shortening the pole intervals. This is very significant, as the side length is 230 m and the intervals can easily be as short as 10 m. With shorter intervals, the large linear extension causes only a limited increase in the final error outcome (equations 11 and 12).

Surveying the ground plan on the construction site

A corner point on the north side is chosen through which a segment of a straight line is drawn oriented to the north. This segment can be extended south from the corner point to determine the position the southern neighboring corner point. This is achieved by sighting along ranging poles set out in sequence and measuring the distance between them with rods aligned along cords stretched between them (fig. 5). The remaining corners are then surveyed and positioned by applying ATMR and EP.

25 The best possible visual acuity of the human eye is 0.4 arc minutes or 24 arc seconds.

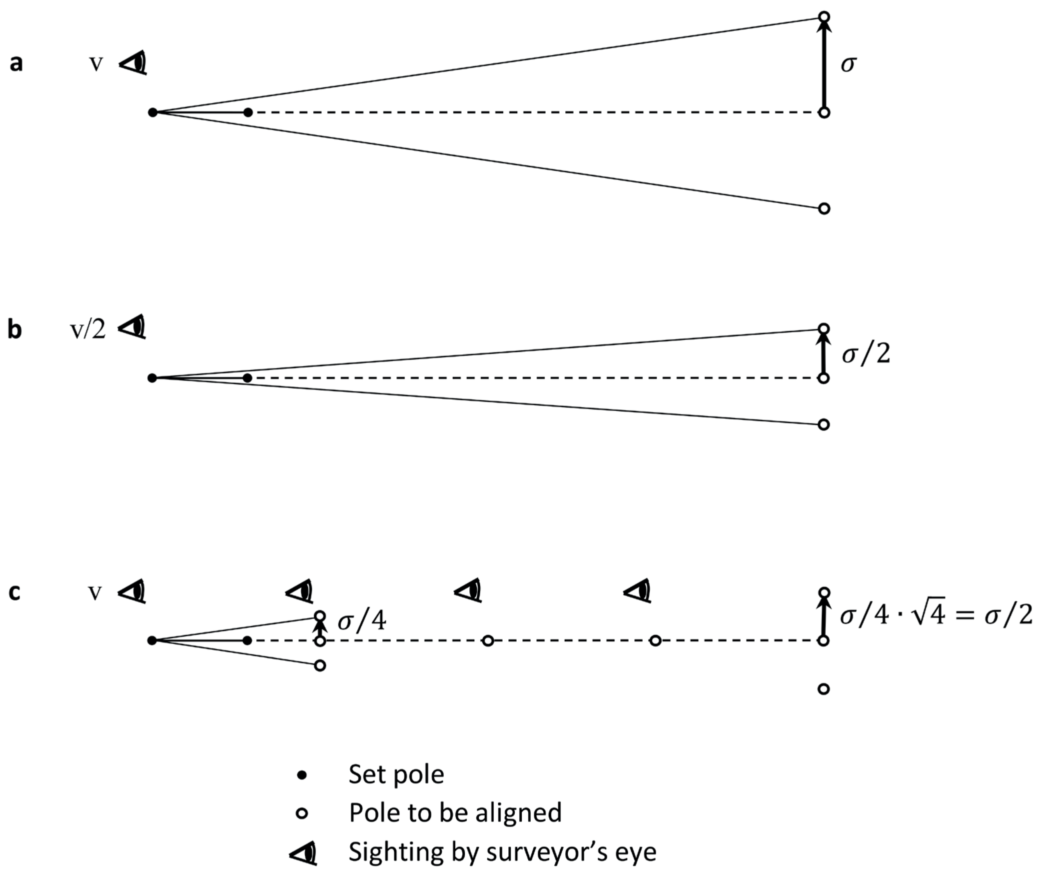

Fig. 7. Increasing the sighting precision by shortening the pole intervals.
a. With a given visual acuity (visual angle v) a single sighting results in an error of σ.
b. Double visual acuity (angle v/2) results in half the error (σ/2).
c. The same half error (σ/2) can be attained with visual acuity v by 4 successive sightings using 1/4 length intervals. The error of a single sighting is σ/4 as it is proportional to the interval size, and this is multiplied by the square root of the number of sightings to obtain the total error value, which is $\sigma/4 \cdot \sqrt{4} = \sigma/2$

Error analysis of the closed setting-out survey

The first surveyed side is an extension of a segment of a straight line aligned to the Northern Celestial Pole (NCP). The setting out of the three remaining perpendiculars with their extensions can start at either of its end points. Thus, both end points of this cardinally aligned side are at the same time starting and end points depending on whether the survey is carried out in a clockwise or anti-clockwise direction. By employing this closed survey strategy any 'misclosure' at these end-points due to error propagation can be detected and the survey repeated until the ground plan's perimeter is closed up. Furthermore, procedures used during the setting-out can also be improved iteratively and the changes evaluated by employing this closed survey strategy.

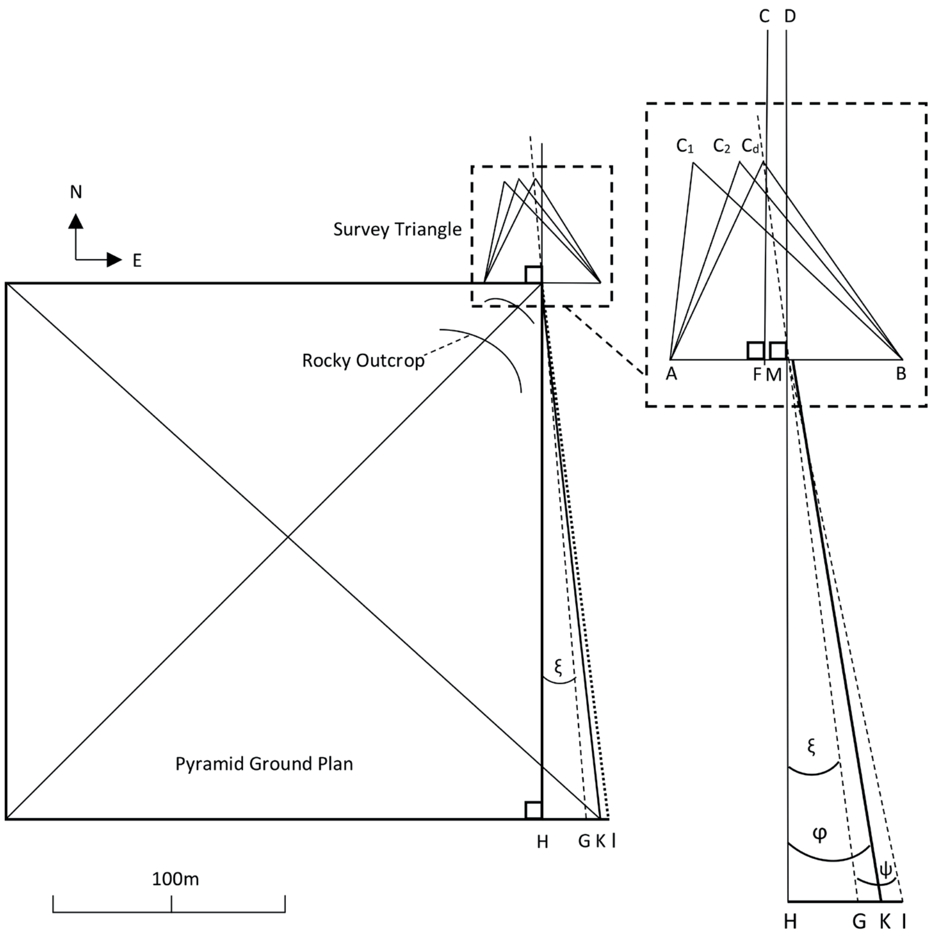

Fig. 8. Deviation produced at the pyramid's corners. This procedure provides an elegant solution that circumvents the rocky outcrop under the center of Khufu's pyramid, but it does introduce a degree of error. $\overline{GH}$ is the deviation produced by projecting an initial line segment constructed as a perpendicular at the corner running through point M and extended until it intersects with the straight line formed by the extension of the opposite side. $\overline{GI}$ is the additional deviation arising from the extension of the constructed perpendicular, which carried an initial error. The total effect of the deviation on the length of the pyramid's side $\overline{HK}$ (φ) arises from superposition during surveying of angular errors $\overline{GH}$ (ξ) and $\overline{GI}$ (ψ) (equation 12). The survey triangle and deviations ξ, ψ, and φ are not to scale.

Discussion

Based on these principles outlined above, an estimate of the real error values can be made in the following section. The angular and length measures are calculated according to the equations in the appendix and based on the precision measuring methods outlined in figs. 5, 6 and 7. For an estimate of the total random error, I assumed a 0.25 mm abutting joint clearance (fig. 6-b), a 0.13 mm layer of fine sand due to the proximity of the dusty desert (grain size 0.063-0.2 mm DIN 4200) (fig. 6-d), 0.2 mm error due to temperature changes of up to 10°C under sun exposure (Eurocode 1, European Standards 1991-1-5, 8-cubit rod = 4.2 m) (fig. 6-e), and 0.0025 mm shortening through deviation from the direction (1 cm horizontally and 1 cm vertically for 8 cubits). Summed up, the segment error is 0.58 mm for 8-cubit long rods or 0.48 mm for 4-cubit long rods.

The choice of measuring tool, either a rod or cord, is of paramount importance. Dorner analyzed the effects of temperature, air moisture, and traction on a weighted hemp cord 8 mm in diameter and about 1 m in length. The values he found extrapolated over 100 cubits (52.4 m) amount to 50 cm for both temperature and air moisture effects and as much as 100 cm under traction. He noted that during the 10-day test, the cord extended by 1 cm, although it had been strained before over several weeks[26] adding another 50 cm over 100 cubits (52.4 m). This level of variation means that it is unlikely that cord-based methods were used for measuring during the pyramid's construction. He considered possible special manufacturing processes for cords and he also considered rods as a long-distance measuring tool[27] but he did not research the latter approach in detail.

Unlike for the dimensionally unstable cords, more detailed computations can be carried out based on wooden rods. As a basis for the computations, I take 8-cubit rods (4.2 m), an error in length for each rod of 0.58 mm, and the surveyor's visual acuity to be 1 arc minute. The calculation of the error produced by the corner survey triangle (fig. 11) and its extension over the pyramid's side (fig. 8) yields, for 40/30 m and 20/30 m base/side triangles respectively, deviations of 3.4 cm (= 30.3 arc seconds) and 4 cm (= 35.6 arc seconds). The latter is an astonishingly close match to the measured mean angular error value of 37 arc seconds.[28] To be practically viable, ATMR had to fulfill 3 more preconditions, namely, allow the free choice of base length, allow different shapes for the isosceles triangles, allow a wide choice of possible apex positions on the ground for point C, and result in a total uncertainty of 60 arc seconds or less. Fig. 9 depicts the huge extent of available triangle sizes and shapes that can be used with 8-cubit (4.2 m) and 4-cubit (2.1 m) rods that meet the requirements and could achieve success.

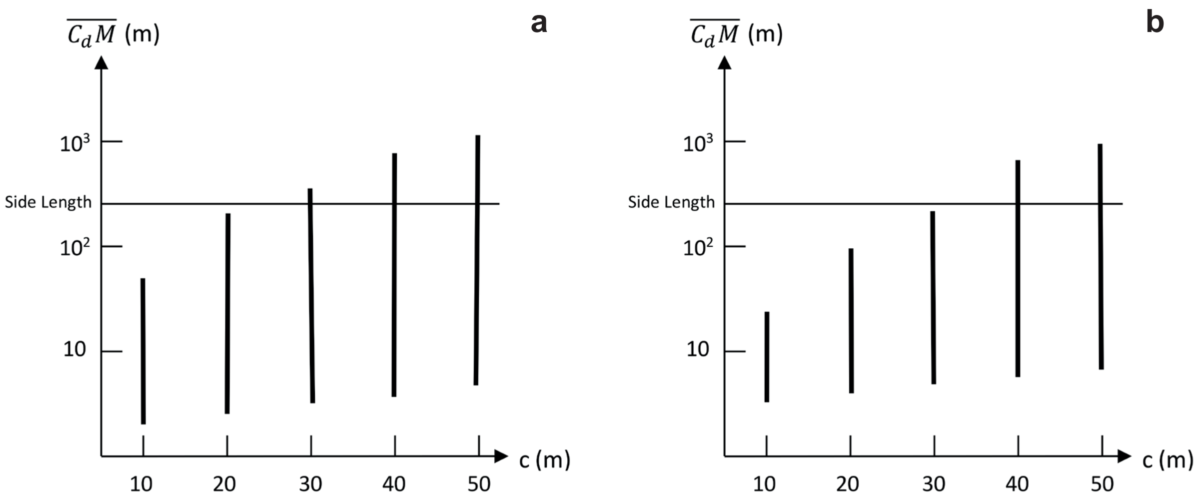

Fig. 9. Viable range of possibilities giving an angular deviation of less than 60 arc seconds. The contribution of the perpendicular's extension to the total error is included. Here, c = base of the isosceles triangle, $\overline{C_dM}$ = height of the isosceles triangle, and side length = 230.36 m (pyramid's side).
a. With 8-cubit rods (1 cubit = 52.44 cm) and an error for each rod of 0.58 mm,
b. With 4-cubit rods and an error for each rod of 0.48 mm.

26 Dorner (1981), pp. 98–99.
27 Dorner (1981), p. 99. Summarized and translated from German by the author.
28 Dorner (1981), p. 76. Arithmetical mean of 16", 19", 55", and 58".

Conclusion

The striking discrepancy between the archaeologically observed values of construction accuracy achieved on the monuments and the lack of evidence of measuring tools or procedures (fig. 6) remains difficult to explain. Based on modern standing-building surveys the ancient surveyors could obtain a deviation due to length measurement error of only 4.3 mm over 230.36 m. In a practical test with measuring sticks, Unterberger was able to reproduce an uncertainty of less than 10 mm over 240 m.[29] In contrast, the differences in the lengths between the northern and western sides as well as between the northern and the eastern sides are 4.4 and 4.1 cm.[30] Spence compared this with the variation in the alignments of the different sides in earlier pyramids.[31] For the Great Pyramid of Giza, the difference in alignment between the western and eastern sides varies by 39 arc seconds,[32] generating a length difference of 4.35 cm. With a 20/30 m base/side survey triangle the ATMR produces 2.6 cm of variation and the EP produces 2.94 cm, which combine to produce a value of 3.93 cm (equation 12 in Appendix). This supports Dorner's proposal[33] and Spence's conjecture that the differences are the result of the construction of right angles at the corners while laying out the ground plan;[34] and it also supports the proposition that the sides were set out and aligned using a 2-step procedure.

After the technical suitability assessment, the historical plausibility of its use must be evaluated in the analysis. What motivated the pharaonic surveyors to develop such an elaborate method, which modern specialists have only now considered and find difficult to replicate? The main technical driver seems to be the disparity between the high accuracy requirement and the low linear dimensional stability of cords. Experiments in field survey have found that rods are far more reliable than cords for producing repeated and accurate measurements. The adoption of methods requiring several sequential steps, however, may have been a derivative of the ancient Egyptians' experience with cut stone construction. In those projects stone blocks of a certain dimension had to be carved to size in many stages, and this sort of iterative approach may have led them to use the ATMR. They could not differentiate between true and approximate values, but the archaeological record represented by the pyramid's base, including the levelled pavement created before the structure's erection is fascinating. "It clearly shows that the survey was executed at least 3 times with increasing accuracy".[35] The difference in height above sea level at the bottom of the corner sockets is 47 cm, while the surface of the leveled bedrock around them varies by 19 cm. The platform on which the pyramid was placed, however, varies only by 2.1 cm.[36] This accuracy is the best confirmation that a succession of approximating procedures were carried out at the appropriate places and at the appropriate times in order to produce the 4 sides and corner angles that can be observed today. When interpreted logically, the available evidence can support a compelling argument that such a technique was used.

Although a small deviation from a true right-angle seems to be a petty detail of ancient Egyptian architecture, the achievement contains more information than is visible at first glance. It allows us a glimpse into the intellectual life of the specialized scribes and architects. It is likely that the Old Kingdom Egyptians used the seqed or seked system to measure slopes, but they did not understand angles in terms of the arc and radius of a circle. The seked system, however, could not measure

29 Unterberger (2008), p. 17.
30 Dorner (1981), p. 77.
31 Spence (2000), p. 321.
32 Dorner (1981), p. 77.
33 Dorner (1981), p. 95.
34 Spence (2000), p. 321.
35 Dorner (1981), p. 151. Translated from German by the author.
36 Borchardt (1937), Heft 1, p. 6.

right angles, as it had to express a rise and a run. The solution they used to produce right angles used the idea of equidistance outlined above. It was simple and could be implemented precisely using only measurement rods and cords. The surveyors would first have chosen a point C_1 fairly close to point C_d (figs. 3 and 4), and by repeating the procedure they would have reached their goal with minor effort. Afterward, they had only to extend the perpendicular along the pyramid's side.

Analysis of the ATMR, with RA values of 0.6 x 10^{-4} and 0.52 x 10^{-4} for 4-cubit and 8-cubit rods respectively, shows that the surveyors could have met or exceeded an accuracy of 60 arc seconds. It is even possible that a surveyor with the very best visual acuity of 0.4 arc minutes could have achieved a value of as little as 25.5 arc seconds of error, and in favorable atmospheric conditions, even down to 14.5 arc seconds. Similarly, the 8 possible arrangements of survey triangles at each corner (fig. 10) would have given the surveyors the freedom to choose the best layout to deal with any variations in the local topography (fig. 8). Stadelmann noted that the precision of the observed measurements can indeed be replicated, but hardly improved upon.[37] The current error analysis shows that the ancient Egyptians had pushed their technology close to its limits.

The most remarkable conclusion is that the use of the ATMR indicates that the ancient Egyptian specialists made use of two concepts formulated in writing only thousands of years later. These concepts are Euclid's definitions of the right angle,[38] and equidistance. Proposition 11, Book 1 [39] reflects the ATMR closely. Equidistance is represented in Definition 1, Book 10 as "*summetria*" (symmetry). It's originally meaning was "in measure with" or "sharing a common measure."[40] "The term quickly acquired a further, more general meaning, that of a proportion relation."[41] This concept was what facilitated the accurate and symmetrical construction of the 4th dynasty pyramids. Its application at the pyramid's corners gives an insight into the state of geometrical knowledge at the time the Great Pyramid's construction began, around 2,554 BC[42] (2,480 BC).[43]

In conclusion, a procedure that combined the ATMR and the EP could well have been the method used to set out the ground plan of Khufu's pyramid tomb. Future research may reveal more arguments for its historicity.

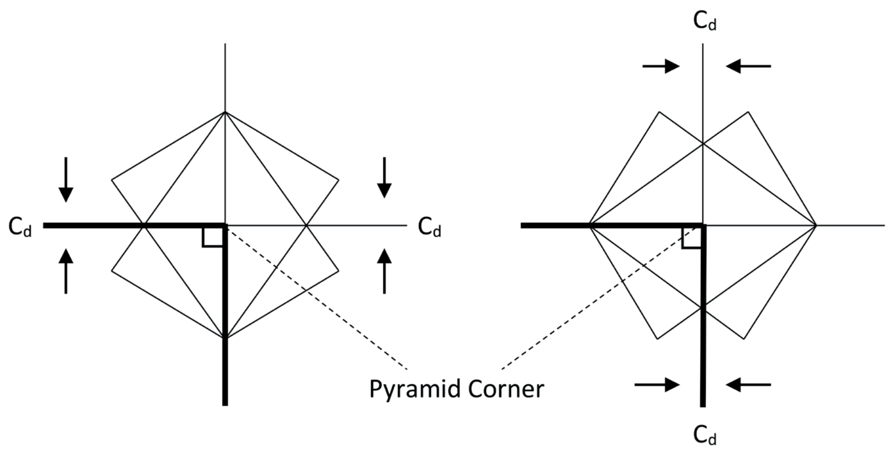

Fig. 10. The 8 possibilities for setting out a survey triangle.

37 Stadelmann (1991), p. 220.
38 Euclid (2007), Elements, Book 1, Definition 10.
39 Euclid (2007), Elements, Book 1, p. 16.
40 Hon, Goldstein (2008), p. 71.
41 Stanford Encyclopedia (Date accessed: 27 February 2020).
42 Von Beckerath (1997), p. 159.
43 Spence (2000), p. 320.

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Appendix

Length uncertainty and angular deviation

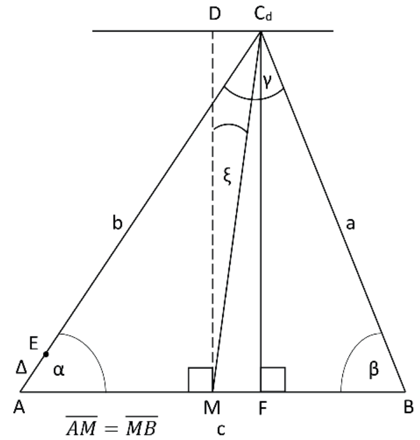

Fig. 11. Uncertainty of the isosceles survey triangle.

$$\overline{C_dD} = \overline{MF} = \overline{AF} - \overline{AM} \quad (1)$$

since $\overline{AF} = b \cdot \cos\alpha$ and $\overline{AM} = \frac{c}{2}$

$$\overline{C_dD} = b \cdot \cos\alpha - \frac{c}{2} \quad (2)$$

as $\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$ law of cosines

$$\overline{C_dD} = b\,\frac{b^2+c^2-a^2}{2bc} - \frac{c}{2} \quad (3)$$

$\Delta = \overline{AE}$ and $b = a + \Delta$ by definition

$$\overline{C_dD} = (a+\Delta)\,\frac{(a+\Delta)^2+c^2-a^2}{2(a+\Delta)c} - \frac{c}{2} \quad (4)$$

rearranged

$$\overline{C_dD} = \Delta\frac{a}{c} + \frac{\Delta^2}{2c} \quad (5)$$

$\frac{\Delta^2}{2c}$ insignificant as $\Delta \ll$ c

$$\overline{C_dD} = \Delta\frac{a}{c} \quad \text{isosceles triangle} \quad (5a)$$

$$\overline{C_dD} = \Delta \quad \text{equilateral triangle as a = c} \quad (5b)$$

Measuring on the construction site

Symbols used in the following equations

L_R = length of measuring rod

L_P = distance (interval) between ranging poles

σ_R = error of single measurement with rods

Σ_R = total error with measuring rods

σ_P = error of single measurement with ranging poles

The error for the whole distance of a leg *a* with $\frac{a}{L_R}$ measurements can be expressed as follows:

$$\Sigma_R = \sigma_R \left(\frac{a}{L_R}\right)^{1/2} \qquad \textbf{(6)}$$

To obtain the deviation $\overline{C_d D}$ on the construction site, the difference Δ between the 2 legs of the isosceles triangle must be calculated. The length of each leg comprises 2 real components, the sum (number) of measuring rods and the accumulated single measuring errors. The rods can simply be counted (measured). In contrast, the normally distributed random errors of unknown magnitude and algebraic sign can only be estimated. Due to the final criterion and the same measuring procedure on both legs, the 2 total errors Σ_R must be assumed to have the same order of magnitude. The difference in the rod components on each leg is 0. To calculate the difference between the normally distributed total errors, the variances (squared deviations) of the variates are taken instead and added up. This is 2 Σ_R^2. [i] The resulting deviation $\sqrt{2}\ \Sigma_R$ is, therefore, the square root and when plugged into equation 5a, it leads to

$$\overline{C_d D} = \sqrt{2}\, \Sigma_R \, \frac{a}{c} \qquad \textbf{(7)}$$

after projecting point M in the direction of the constructed perpendicular $\overline{MC_d}$ onto the opposite side, with

$\overline{BF} = \frac{c}{2}$ as $\overline{MF} << \overline{BM}$ and

$$\overline{C_d F} = \left(a^2 - \frac{c^2}{4}\right)^{1/2} \qquad \textbf{(8)}$$

i Weisstein (2020). Amazingly, the distribution of a difference or sum of 2 normally distributed independent variates X and Y with means μ_X and μ_Y and variances σ_X^2 and σ_Y^2 is another normal distribution, with mean $\mu_{X-Y} = \mu_X - \mu_Y$ for the difference and $\mu_{X+Y} = \mu_X + \mu_Y$ for the sum, and variance $\sigma^2_{X-Y} = \sigma^2_X + \sigma^2_Y$. In the present case, μ corresponds to the length measured with rods and σ^2 to the squared total deviations Σ_R. X and Y are the 2 isosceles legs of the isosceles triangle.

Symbols pertaining to deviations at the pyramid corner (Fig. 8).

$\overline{GH}$ = deviation at corner contributed through the projection of M in the direction of the constructed perpendicular onto the straight line of the opposite side.

$\overline{GI}$ = deviation at corner contributed through the extension of the perpendicular

$\overline{HK}$ = total deviation due to superposition in practice (equation 12).

$\overline{GH} = \overline{C_d D} \cdot \frac{\overline{HM}}{\overline{C_d F}}$ The deviation at the corner is **(9)**

$$\overline{GH} = \sqrt{2}\, \Sigma_R \, \frac{a}{c} \, \frac{\overline{HM}}{\overline{C_d F}} \quad \mathbf{(10)}$$

To determine the neighboring corner, the constructed perpendicular must be extended over the pyramid's side. This is done by extending the constructed perpendicular by setting out a series of straight lines with ranging poles (Fig. 7.) at intervals of 45 m, for instance. With a surveyor's visual acuity of 1 arc minute (Snellen 20/20), an error σ_P of 1.3 cm arises at each measurement leading to the following:

$$\overline{GI} = \sigma_P \left(\frac{\overline{GM}}{L_P} \right)^{1/2} \quad \mathbf{(11)}$$

The sum of the deviations is calculated according to a normal sum distribution by taking the square root of the sum of the variances (= squared deviations).

$$\overline{HK} = \left(\overline{GH}^2 + \overline{GI}^2 \right)^{1/2} \quad \mathbf{(12)}$$

[ii]

[ii] Weisstein (2000), see footnote i. In the present case, the angular means are both 0. The constructed perpendicular deviates only by random errors from the true 90-degree angle and the extension is in the same direction as the constructed perpendicular.