- VI Jaen Conference on Approximation Theory
- Úbeda, Jaén, Spain, June 28th-July 3rd, 2015
- Approximation Theory, Computer Aided Geometric Design, Numerical Methods and Applications

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# Introduction

This Conference is an activity of the Jaen Approximation Project. Jaen Approximation Project has organized ten editions of the Ubeda Meeting on Approximation and five editions of the Jaen Conference on Approximation. It also issues the Jaen Journal on Approximation since 2009.

The objective of these conferences is to provide a useful and nice forum for researchers in the subjects to meet and discuss. In this sense, the conference program has been designed to keep joined the group during four days with a program full of scientific and social activities.

The Conference will be devoted to some significant aspects on Approximation Theory, Computer Aided Geometric Design, Numerical Methods and the Applications of these fields in other areas.

It features seven invited speakers (Asuman G. Aksoy, Annie Cuyt, Kathy Driver, Dany Leviatan, Carla Manni, Gerlind Plonka and Maria Skopina) who will give 50 minutes plenary lectures. Researchers were invited to contribute with a talk or a poster. We have scheduled 16 talks and a poster session.

The Conference is held in Úbeda, what gives participants the opportunity to visit World Heritage Sites and taste a wide culinary variety.

We hope that you all enjoy the Conference, both participants and accompanying people. We are grateful to all those who have made this project a reality; the University of Jaén (Vicerrectorado de Investigación and Departamento de Matemáticas), Diputación Provincial de Jaén, Ayuntamiento de Úbeda and UNED. Here we emphasize our commitment to keep on working to improve our university and our province.

*The Organizing Committee*

# Scientific Program

| | **June, 28th-Sunday** |
|---|---|
| **21:30-** | **Dinner**<br>(Parador de Úbeda) |

| | **June, 29th-Monday** | |
|---|---|---|
| **9:00-** | **Summer solstice**<br>(Sinagoga del agua) | |
| **10:30-11:15** | **OPENING CEREMONY** | UNED (Úbeda) |
| **11:15-11:40** | **Coffee Break** | |
| | **SESSION 1**<br>(Chairperson D. Leviatan) | |
| **11:40-12:30** | Annie Cuyt (p. 5) | |
| **12:30-12:55** | Christophe Rabut (p. 40) | |
| **12:55-13:20** | O. Mula (p. 26) | |
| **13:20-13:45** | Heinz-Joachim Rack (p. 41) | |
| **13:45-14:10** | Georg Zimmermann (p. 47) | |
| **14:30-** | **Lunch**<br>(Hotel María de Molina) | |
| **20:00-** | **Visit to Baeza** | |
| **21:00-** | **Cocktail** | |

| | **June, 30th-Tuesday** | |
|---|---|---|
| | **SESSION 2** (Chairperson A. Cuyt) | UNED (Úbeda) |
| **10:00-10:50** | Asuman Güven Aksoy (p. 3) | |
| **10:50-11:15** | J. M. Almira (p. 15) | |
| **11:15-11:40** | **Coffee Break** | |
| | **SESSION 3** (Chairperson G. Mastroianni) | |
| **11:40-12:30** | Gerlind Plonka (p. 10) | |
| **12:30-12:55** | M. A. Piñar (p. 38) | |
| **12:55-13:20** | Laurent Gajny (p. 20) | |
| **13:20-13:45** | A. M. Delgado (p. 18) | |
| **13:45-14:10** | J. Szabados (p. 46) | |
| **14:30-** | **Lunch** (Parador de Úbeda) | |
| **19:00-21:30** | **Visit to Úbeda** | |
| **21:30-** | **Gala Dinner** (Hospital de Santiago) | |

| | **July, 1st-Wednesday** | |
|---|---|---|
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| **10:00-10:50** | Maria Skopina (p. 11) | |
| **10:50-11:15** | Peter Binev (p. 17) | |
| **11:15-11:40** | **Coffee Break** | |
| | **SESSION 5** (Chairperson J. Szabados) | |
| **11:40-12:30** | Dany Leviatan (p. 7) | |
| **12:30-12:55** | Hare Krishna Nigam (p. 28) | |
| **12:55-13:20** | I. Notarangelo (p. 29) | |
| **13:20-13:45** | G. Mastroianni (p. 24) | |
| **13:45-14:10** | M.A. Jiménez (p. 22) | |
| **14:30-** | **Lunch** (Hotel María de Molina) | |
| **19:00-** | **Visit to Cazorla** | |
| | **Dinner** (Cazorla) | |

| | **July, 2nd-Thursday** | |
|---|---|---|
| | **SESSION 6** (Chairperson M. Skopina) | UNED (Úbeda) |
| **10:00-10:50** | Kathy Driver (p. 6) | |
| **10:50-11:15** | **Poster Session** | |
| **11:15-11:40** | **Coffee Break** | |
| | **SESSION 6** (Chairperson M.A. Jiménez) | |
| **11:40-12:30** | Carla Manni (p. 8) | |
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| **13:30-** | **Closure Ceremony** | |
| **14:00-** | **Lunch** (Hotel María de Molina) | |
| **19:00-** | **Visit to Torreón** **Visit to San Lorenzo** | |
| **21:00-** | **Dinner** (Parador de Úbeda) | |

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- Teresa E. Pérez (p. 36)

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## Invited Lectures



## Short Talks/Posters

# Invited Lectures

# Bernstein's lethargy theorem in Fréchet spaces

Asuman Güven Aksoy and Grzegorz Lewicki

**Abstract**

In this talk, we consider Bernstein's Lethargy Theorem (BLT) [3] in the context of Fréchet spaces. Let $X$ be an infinite-dimensional Fréchet space and let $\mathcal{V} = \{V_n\}$ be a nested sequence of subspaces of $X$ such that $\overline{V_n} \subseteq V_{n+1}$ for any $n \in \mathbb{N}$ and $X = \overline{\bigcup_{n=1}^{\infty} V_n}$. Let $e_n$ be a decreasing sequence of positive numbers tending to 0. Under an additional natural condition on $\sup\{\mathrm{dist}(x, V_n)\}$, we prove that, there exists $x \in X$ and $n_o \in \mathbb{N}$ such that

$$\frac{e_n}{3} \leq \mathrm{dist}(x, V_n) \leq 3e_n$$

for any $n \geq n_o$. By using the above theorem, we prove both Shapiro's [6], [1] and Tyuremskikh's [7] theorems for Fréchet spaces. Considering rapidly decreasing sequences, other versions of the BLT theorem [4] in Fréchet spaces will be discussed. We also give a theorem improving Konyagin's [5] result for Banach spaces.

**Keywords:** best approximation, Bernstein's lethargy theorem, Fréchet spaces.
**AMS Classification:** 41A25, 41A50, 41A65.

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# Exponential analysis, Sparse interpolation and Padé approximation

Annie Cuyt and Wen-shin Lee

**Abstract**

A common underlying problem statement in many applications is that of determining the number of components, and for each component the value of the frequency, damping factor, amplitude and phase in a multi-exponential model. It occurs, for instance, in magnetic resonance and infrared spectroscopy, vibration analysis, seismic data analysis, electronic odour recognition, keystroke eavesdropping, nuclear science, music signal processing, transient detection, motor fault diagnosis, electrophysiology, drug clearance monitoring and glucose tolerance testing, to name just a few. The general technique of multi-exponential modeling is closely related to what is commonly known as the Padé-Laplace method in approximation theory, and the technique of sparse interpolation in the field of computer algebra. The problem of multi-exponential modeling is an inverse problem and therefore may be severely ill-posed, depending on the relative location of the frequencies and phases. Besides the reliability of the estimated parameters, the sparsity of the multi-exponential representation has also become important. A representation is called sparse if it is a combination of only a few elements instead of all available generating elements. In sparse interpolation, the aim is to determine all the parameters from only a small amount of data samples, and with a complexity proportional to the number of terms in the representation. Despite the close connections between these fields, there is a clear lack of communication and cross-fertilization in the scientific literature. We present the basics of multi-exponential modelling, connect the problem to sparse interpolation and show how to improve the technique using results from Padé approximation theory: the conditioning is improved, the parameter detection is validated and the convergence of the method is accelerated. The new algorithm is applied to a number of challenging applications.

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# Orthogonal polynomials and interlacing of zeros*

Kathy Driver, Martin E. Muldoon and Kerstin Jordaan

**Abstract**

The interlacing of zeros of two polynomials of consecutive degree in an orthogonal sequence is a classical result that has important applications to Gauss quadrature. Stieltjes extended the concept of interlacing to zeros of two orthogonal polynomials of non-consecutive degree. In this more general context, common zeros of the two polynomials involved (if there are any) play a critical role when interlacing of zeros is under consideration.

This talk will focus on sequences of Laguerre polynomials $\{L_n^{(\alpha)}\}_{n=0}^{\infty}$, $\alpha$ fixed, $\alpha > -1$. Recent developments on the interlacing of zeros of orthogonal polynomials from different sequences within this classical family will be discussed and the connection between the interlacing of zeros and the existence of common zeros will be highlighted. The mixed three term recurrence relations satisfied by Laguerre polynomials corresponding to different values of the parameter $\alpha$ are used to derive bounds for the largest and smallest zeros of Laguerre polynomials. We discuss the interlacing of zeros, and the co-primality, of the quasi-orthogonal Laguerre sequences $\{L_n^{(\alpha)}\}_{n=0}^{\infty}$ for $\alpha$ fixed, $-2 < \alpha < -1$.

**Keywords:** special functions and approximation.
**AMS Classification:** 41A28, 41A40, 41A60.

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# Comparing the degrees of unconstrained and constrained approximation by polynomials

D. Leviatan

**Abstract**

It is quite obvious that one should expect that the degree of constrained approximation be worse than the degree of unconstrained approximation. However, it turns out that in certain cases we can deduce the behavior of the degrees of the former from information about the latter.

Let $E_n(f)$ denote the degree of approximation of $f \in C[-1,1]$, by algebraic polynomials of degree $< n$, and assume that we know that for some $\alpha > 0$ and $\mathcal{N} \geq 1$,

$$n^\alpha E_n(f) \leq 1, \quad n \geq \mathcal{N}.$$

Suppose that $f \in C[-1,1]$, changes its monotonicity or convexity $s \geq 0$ times in $[-1,1]$ ($s = 0$ means that $f$ is monotone or convex, respectively). We are interested in what may be said about its degree of approximation by polynomials of degree $< n$ that are comonotone or coconvex with $f$. Specifically, if $f$ changes its monotonicity or convexity at $Y_s := \{y_1, \dots, y_s\}$ ($Y_0 = \emptyset$) and the degrees of comonotone and coconvex approximation are denoted by $E_n^{(q)}(f, Y_s)$, $q = 1, 2$, respectively. We investigate when can one say that

$$n^\alpha E_n^{(q)}(f, Y_s) \leq c(\alpha, s, \mathcal{N}), \quad n \geq \mathcal{N}^*,$$

for some $\mathcal{N}^*$. Clearly, $\mathcal{N}^*$, if it exists at all (we prove it always does), depends on $\alpha$, $s$ and $\mathcal{N}$. However, it turns out that for certain values of $\alpha$, $s$ and $\mathcal{N}$, $\mathcal{N}^*$ depends also on $Y_s$, and in some cases even on $f$ itself, and this dependence is essential.

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# Isogeometric methods based in generalized B-splines

Carla Manni

**Abstract**

Isogeometric analysis (IgA) is a well-established paradigm for the analysis of problems governed by partial differential equations (PDEs). It provides a design-though-analysis connection by exploiting a common representation model. This connections is achieved by using the functions adopted in Computer Aided Design (CAD) systems not only to describe the domain geometry, but also to represent the numerical solution of the differential problem. CAD software, used in industry for geometric modeling, typically describes physical domains by means of tensor-product B-splines and their rational extension, the so-called NURBS. In its original formulation IgA is based on the same set of functions, see [1,2].

Nonetheless, the IgA paradigm is not confined to B-splines, NURBS and their localized extensions. Other possible discretization techniques have also received some attention; among the others, we mention generalized splines [3].

The so-called generalized B-splines (GB-splines) are piecewise functions with sections in more general spaces than algebraic polynomial spaces (like classical B-splines). Suitable selections of such spaces –typically including trigonometric or exponential functions– allow an exact representation of polynomial curves, conic sections, helices and other profiles of salient interest in applications. GB-splines possess all fundamental properties of algebraic B-splines: recurrence relation, compact minimum support, local linear independence, (non-stationary) subdivision rule, etc. Moreover, contrarily to rational extensions like NURBS, they behave completely similar to B-splines with respect to differentiation and integration, see [3] and references therein. Finally, GB-splines support (locally refined) hierarchical structures the same way as classical polynomial B-splines [4].

Tensor-product GB-splines and their hierarchical counterpart have been used in IgA following the Galerkin or collocation formulation, see [3,4,5] and references therein. Thanks to their complete structural similarity with classical B-splines (which is based on a Bernstein-like representation), GB-splines are plug-to-plug compatible with B-splines in IgA. On the other hand, when dealing with GB-splines, the section spaces can be selected according to a problem-oriented strategy taking into account the geometrical and/or analytical peculiar issues of the specific addressed problem. The finite-tuning of the approximation spaces generally results in a gain from the accuracy point of view.

In this talk we review some isogeometric methods based on trigonometric and exponential generalized spline spaces for their relevance in practical applications.

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# Deterministic sparse FFT algorithms*

Gerlind Plonka

**Abstract**

We consider some ideas to improve the well-known (inverse) FFT algorithm to compute a vector $\mathbf{x} \in \mathbb{C}^N$ from its Fourier transformed data. It is known that the FFT of length $N$ needs $O(N \log N)$ arithmetical operations. However, if the resulting vector $\mathbf{x}$ is a-priori known to be sparse, i.e., contains only a small number of non-zero components, the question arises, whether we can do this computation in an even faster way. In recent years, different sublinear algorithms for the sparse FFT have been proposed, most of them are randomized. We want to concentrate on deterministic sparse FFT algorithms and consider especially vectors with short support and sparse positive vectors. The talk is based on joint work with Manfred Tasche and Katrin Wannenwetsch.

**Keywords:** discrete Fourier transform, sparse Fourier reconstruction, sparse FFT.
**AMS Classification:** 65T50, 42A38.

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# Exact and Falsified Sampling Approximation

Maria Skopina

**Abstract**

We study approximation properties of the expansions $\sum_{k\in\mathbb{Z}^d} c_k\varphi(M^j x + k)$, where $M$ is a matrix dilation, $c_k$ is either the sampled value of a function $f$ at $M^{-j}k$ or its integral average near $M^{-j}k$ (falsified sampled value). Error estimations in $L_p$-norm, $2 \le p \le \infty$, are given in terms of the Fourier transform of $f$. The approximation order depends on the decay of $\widehat{f}$ and on the order of Strang-Fix condition for $\phi$. The estimates are obtained for a wide class of $\varphi$ including both compactly supported and band-limited functions. The band-limited functions $\varphi$ provide an arbitrarily large approximation order, while the compactly supported functions are more preferable for implementations. For the one-dimensional case, we also constructed sampling wavelet decompositions, i.e. frame-like wavelet expansions with coefficients interpolating a signal $f$ at the dyadic points.

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# Short Talks/Posters

# Some remarks on the closure of translation-dilation invariant linear spaces of polynomials

J. M. Almira and L. Székelyhidi

**Abstract**

At the 49th International Symposium on Functional Equations in Graz, Mariatrost, Austria, 2011 and later at the 14th International Conference on Functional Equations and Inequalities in Będlewo, Poland, 2011 the second author proposed the following problem: Assume that $V$ is a linear space of real polynomials in $n$ variables which is translation invariant. Suppose moreover that the sequence $(p_n)$ in $V$ converges pointwise to a polynomial $p$. Is it true that $p$ is in $V$? Despite several efforts of different researchers this question has still remained open. In this note we solve the problem in the positive for the special case when $V$ is a translation-dilation invariant linear space of polynomials.

**Keywords:** polynomials, translation invariance, dilation invariance, pointwise convergence, difference operators.
**AMS Classification:** primary 41A10, 40A30; secondary 39A70.

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[3] **L. Székelyhidi**, *Problem 4*, Problems and Remarks section of the 14th International Conference on Functional Equations and Inequalities, Będlewo, September 11–17, 2011.

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# Near-optimal adaptive approximations*

Peter Binev

**Abstract**

We consider approximation algorithms that are based on a posteriori information about the local errors. Given a budget $N$ of total number of degrees of freedom, we want to build a partition of the domain and assignments of a number of degrees of freedom for each element of this partition in such a way that the resulting error of approximation will be comparable with the best possible one. In Finite Element Methods this type of adaptive approximation is known as *h-adaptive*, in case the numbers of degrees of freedom for each element of the partition is the same, or *hp-adaptive* if the number of degrees of freedom may vary. We present algorithms in a very general setup for both the $h$-adaptive and $hp$-adaptive cases and prove that they provide a near-best approximation in a sense that the error and the number of degrees of freedom are both within a multiplicative constants from the best possible approximation that uses full knowledge about the function. The results of this talk will appear in [1].

**Keywords:** adaptive algorithms, finite element methods, h-adaptivity, hp-adaptivity, tree-based algorithms, near-best approximation, instance optimality.

**AMS Classification:** 41A15, 41A63, 65D15, 65M55, 68Q32, 68W25, 97N50.

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# Orthogonal polynomials on the unit ball*

A. M. Delgado, L. Fernández, T. E. Pérez and M. A. Piñar

**Abstract**

In this talk we present a study of the Multivariate Sobolev Orthogonal Polynomials with respect to the Sobolev inner product on the unit ball given by

$$\langle f, g\rangle_\mu^S = \frac{1}{\omega_\mu}\int_{\mathbb{B}^d} f(x)g(x)(1-|x|^2)^\mu dx + \frac{\lambda}{\sigma_d}\int_{\mathbb{S}^{d-1}} \frac{\partial f}{\partial \mathbf{n}}(\xi)\frac{\partial g}{\partial \mathbf{n}}(\xi)d\sigma(\xi),$$

where $\mu > -1$, $\lambda > 0$, $d\sigma(\xi)$ denotes the surface measure on the unit sphere $\mathbb{S}^{d-1}$, $\sigma_d$ and $\omega_\mu$ are normalizing constants, and $\frac{\partial}{\partial \mathbf{n}}$ stands for the outward normal derivative operator. An explicit exppresion in terms of univariate Sobolev orthogonal polynomials and spherical harmonics is given. Using this explicit exppresion, some properties of the polynomials, kernel functions and Christoffel functions are explored.

**Keywords:** multivariate orthogonal polynomials, normal derivative, kernel function, Christoffel function.

**AMS Classification:** 33C50, 42C10.

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# Best $L_1$ approximation and locally computed $L_1$ spline fits of the Heaviside function and multiscale univariate datasets

Laurent Gajny, Olivier Gibaru and Eric Nyiri

**Abstract**

Best $L_1$ approximations of the Heaviside function in Chebyshev and weak-Chebyshev spaces has a Gibbs phenomenon. It has been shown in the nineties for the trigonometric polynomial [1] and polygonal line cases [2]. By mean of recent results of characterization of best $L_1$ approximation in Chebyshev and weak-Chebyshev spaces [3] that we recall, this Gibbs phenomenon can also be evidenced in the polynomial and polynomial spline cases. It can be reduced in this latter case by using $L_1$ spline fits [4] which are best $L_1$ approximations in an appropriate spline space obtained by the reunion of $L_1$ interpolation splines [5]. These splines are known to preserve the shape of the Heaviside function [6]. We prove here the existence of $L_1$ spline fits. Their major disadvantage is that obtaining them can be time consuming. Thus we propose a sliding window method on seven knots which is as efficient as the global method but within a linear complexity on the number of spline knots. This algorithm can also be fairly applied to the problem of approximation of datasets with abrupt changes of magnitude.

**Keywords:** $L_1$ norm, shape preserving approximation, polynomial spline, Heaviside function.

**AMS Classification:** 41A10, 41A15, 41A50, 41A52.

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# A note on the existence of certain extremal Lipschitz functions

Miguel A. Jiménez-Pozo

**Abstract**

For every $0 < \beta < \alpha \leq 1$, the following inclusions hold strictly,

$$\mathrm{lip}_1 \subset C' \subset D \subset \mathrm{Lip}_\alpha \subset \mathrm{lip}_\beta \subset C \subset B,$$

where the symbols denote the traditional linear spaces of real valued $2\pi$–periodic constants, differentiable, Lipschitz, continuous or bounded functions. As known, there exists a function $f \in C[0, 2\pi]$ that is not differentiable at any point of its domain. We refine this result by proving the existence of a function $f \in \mathrm{lip}_\beta[0, 2\pi]$ that is not locally in $\mathrm{lip}_\beta$ at any point.

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# Orthogonal polynomials on the ball with an extra term on the sphere*

Clotilde Martínez and Miguel A. Piñar

**Abstract**

In this work we study a family of mutually orthogonal polynomials on the unit ball with respect to an inner product which includes an additional term on the sphere. First, we will get connection formulas relating classical multivariate orthogonal polynomials on the ball with our family of orthogonal polynomials. Then, using the representation of these polynomials in terms of spherical harmonics differential properties will be deduced.

**Keywords:** orthogonal polynomials in several variables, partial differential equations.
**AMS Classification:** 42C05, 33C50.

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# Some new interpolation processes

G. Mastroianni and I. Notarangelo

**Abstract**

The polynomial approximation of functions defined on the real semiaxis and having exponential growth at the endpoints has received little attention in the literature. Only recently, in [2] and [1], some weighted polynomial inequalities and estimates for the best weighted approximation, with suitable moduli of smoothness and related $K-$functionals, have been proved.

In this talk we will introduce a new interpolation process for these classes of functions, showing that it converges with the order of the best approximation in weighted $L^p-$metric, $1 < p < \infty$.

**Keywords:** Weighted polynomial approximation, exponential weights, unbounded interval, real semiaxis, Lagrange interpolation.
**AMS Classification:** 41A10, 41A05.

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# Approximation strategies to couple data assimilation with model-driven simulations*

O. Mula

**Abstract**

The accurate approximation of physical systems is a central task in numerous branches of science and engineering. It can be addressed through two main paradigms that have traditionally been studied separately: data-driven and model-based methods. A rather recent trend is to couple both methodologies so as to correct deficient or inaccurate models through measurements or to regularize estimation problems through a model. Although some aspects of this topic are covered by uncertainty quantification, the works that we would like to present in this talk have a somewhat different flavor. Here is a brief summary of the contents.

Our general setting is the following. Let $H$ be a Hilbert space of functions over a domain $D \in \mathbb{R}^p$, $p \in N^*$. We have a function $u \in H$ that we want to approximate. This function is a field of interest of a physical system taking place in $D$. The tools for our reconstruction are the knowledge that $u$ belongs to some compact set $\mathcal{M} \subset H$ and the knowledge of some measurements of $u$, that we denote $\lambda_i(u)$, $i = 1, ..., M$ in the form of linear functionals applied to $u$. For example, we can assume that $u$ is the solution of a parameter dependent PDE, where we do not know the parameters. In this case $\mathcal{M}$ is the family of solutions to the PDE when the parameters vary. The $M$ linear functionals come from a dictionary $\Sigma$ of the dual of $H$.

The problem has so far been addressed (see [1, 2, 3, 4]) by searching (in a greedy fashion) suitable $N$ dimensional spaces $X_N = \{\varphi_i \in \mathcal{X}\}_{i=1}^N$ so that any $u \in \mathcal{M}$ is approximated by

$$u \approx \sum_{j=1}^{N} c_j(u)\varphi_j,$$

where, for $1 \leq j \leq N$, $c_j(u)$ is a linear combination of $\lambda_i(u)$, $1 \leq i \leq M$. The $M$ "most suitable" linear functionals of $\Sigma$ are also traditionally searched in a greedy fashion.

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One of the main issues that the already existing approaches present is that the behavior with $N$ of the norm of the approximation operators is not properly understood. It is also well-known that cases in which it blows up could occur, which would dramatically degrade the quality of approximation. In our talk, after recalling more in detail this issue, we will present an ongoing work with Prof. Binev in which we build a new approximation strategy based on projection that avoids the presence of this type of uncontrolled terms in the error estimates and provides, in turn, more robustness. We will see that our procedure involves a greedy algorithm whose analysis presents some new challenges with respect to other settings in which greedy algorithms are classically employed.

**Keywords:** data assimilation, reduced basis, greedy algorithm, generalized empirical interpolation, PBDWF.
**AMS Classification:** 41A25, 41A45.

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# Degree of approximation of a function and a conjugate function belonging to the Lip$(\alpha, r)$ class by $(E, q)(C, 1)$ product means

Hare Krishna Nigam

**Abstract**

In the present paper, two new theorems on degree of approximation of a function $f$ and its conjugate function $\bar{f}$, belonging to the Lip$(\alpha, r)$ class by $(E, q)(C, 1)$ product means are established.

**Keywords:** degree of approximation, Lip$(\alpha, r)$ class of function, $(E, q)(C, 1)$ product means, Fourier series, conjugate Fourier series, Lebesgue integral.

**AMS Classification:** 42B05, 42B08.

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# Lagrange–Hermite interpolation processes on the real semiaxis*

G. Mastroianni, I. Notarangelo and P. Pastore

**Abstract**

In this talk we discuss the weighted polynomial interpolation of continuous functions on $[0, +\infty)$, which are $(r-1)-$times differentiable at 0 and can increase with order $\mathcal{O}\left(e^{x^\beta/2}\right)$, $\beta > 1/2$, for $x \to \infty$.

The presence of the derivatives of the function in 0 leads in a natural way to the construction of Lagrange–Hermite polynomial $\mathcal{L}_{m,r}(w,f)$ based at generalized Laguerre zeros, 0 as a multiple node and another additional node (see also [3]). Applying the operator $\mathcal{L}_{m,r}(w)$ to a suitable finite section of the function $f$, we obtain a new interpolation process, that we will denote by $\mathcal{L}^*_{m,r}(w)$ (see [1]).

This new operator is not a projector into the set of all polynomials of degree at most $m+r$, $\mathbb{P}_{m+r}$, but on a special subset $\mathcal{P}^*_{m,r} \subset \mathbb{P}_{m+r}$, where $\bigcup_m \mathcal{P}^*_{m,r}$ is dense in weighted $L^p-$ spaces. We show that, under proper necessary and sufficient conditions, this new interpolation process converges in weighted $L^p-$metric, $1 < p < \infty$, with the order of the best weighted polynomial approximation (that can be found in [2]).

**Keywords:** weighted polynomial approximation, Hermite–Lagrange interpolation, generalized Laguerre weights, unbounded interval, real semiaxis.

**AMS Classification:** 41A10, 41A05.

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# Hole filling splines with volumen constrains by radial basis functions*

M. Pasadas, M.A. Fortes, P. González and A. Palomares

**Abstract**

In many situations we have to fill one or several holes of certain function defined in a domain where there is a lack of information inside some sub-domains. For this we have developped some methods (see [1, 2, 3, 4]).

But in some practical cases we just know some specific geometrical constraints, of industrial or design type, as the special case of a specified volume inside each one of these sub-domains. In this work we study this particular issue, giving both some theoretical and computational results that assures the feasibility of the corresponding procedures.

The studied method in this work manage to find a function of a vector space generated by a radial function basis that minimizes certain quadratic functional that includes some terms associated with the volume constrain and the usual semi-norms in a Sobolev space. In this way, some approximation methods have been developed (see [5, 6]).

In next Section 2 we establish some general and specific notation as the functional spaces where we obtain the reconstructed functions. In section 3 we pose the problem of finding a function that fill a given hole and fulfils a volume restriction . In Section 4 we establish the computation algorithm and a convergence result.

## §1. Preliminaries and notation

Let $m \geq 1$ be a positive integer and let $\Pi_{m-1}(\mathbb{R}^2)$ denote the space of polynomials on $\mathbb{R}^2$ of degree at most $m-1$ whose dimension is $d(m) = \dfrac{m(m+1)}{2}$. Let $\{q_1, \ldots, q_{d(m)}\}$ the standard basis of $\Pi_{m-1}(\mathbb{R}^2)$ .

Consider the following function

$$\phi_\varepsilon(t) = -\frac{1}{2\varepsilon^3}\left(e^{-\varepsilon\sqrt{t}} + \varepsilon\sqrt{t}\right), \; \varepsilon \in \mathbb{R}^+$$

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and the following radial function

$$\Phi_\varepsilon(\mathbf{x}) = \phi_\varepsilon(\langle \mathbf{x}\rangle_2^2) = -\frac{1}{2\varepsilon^3}\left(e^{-\varepsilon\langle \mathbf{x}\rangle_2} + \varepsilon\langle \mathbf{x}\rangle_2\right), \ \mathbf{x} \in \mathbb{R}^2,$$

where $\langle \cdot \rangle_k$ is the Euclidean norm in $\mathbb{R}^k$.

Let $\Omega$ be an open bounded nonempty subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary.

We will use the classical notation $H^k(\Omega)$ to denote the usual Sobolev space of all distributions $u$ which all of whose derivatives up to and including order $k$ are in the classical Lebesgue space $L^2(\Omega)$.

The Sobolev space $H^k(\Omega)$ is a Hilbert space equipped with the inner semi-products given by

$$(u, v)_\ell = \sum_{|\alpha|=\ell} \int_\Omega D^\alpha u(\mathbf{x}) D^\alpha v(\mathbf{x}) d, \ 0 \le \ell \le k,$$

the semi-norms given by $|u|_\ell = (u, u)_\ell^{\frac{1}{2}}$, for all $\ell = 0, \dots, k$, and the norm $\|u\|_k = \left(\sum_{\ell \le k} |u|_\ell^2\right)^{\frac{1}{2}}$.

## §2. Hole filling meshfree smoothing spline surface with volume constraint

For any $N \ge d(m)$, let us an arbitrary set $\mathcal{A}^N = \{\mathbf{a}_1, \dots, \mathbf{a}_N\} \subset \mathbb{R}^2$ such that it contains a $\Pi_{m-1}(\mathbb{R}^2)$-unisolvent subset (i.e., if $q \in \Pi_{m-1}(\mathbb{R}^2)$ and for all $\mathbf{a} \in \mathcal{A}^N$, $q(\mathbf{a}) = 0$ then $q = 0$).

Let $h = h(N)$ be the fill-distance from $\mathcal{A}^N$ to $\Omega$ defined by

$$h = \sup_{\mathbf{x}\in\overline{\Omega}} \inf_{\mathbf{a}\in\mathcal{A}^N} \langle \mathbf{x} - \mathbf{a}\rangle_2$$

and suppose that

$$h = o(N), \ N \to +\infty. \tag{1}$$

On other hand, for any $M \ge N$ let $H^M$ (the hole) be an open nonempty subset of $\Omega$ verifying

$$\mu(H^M) = o(1), \ M \to +\infty, \tag{2}$$

where $\mu$ represents the Lebesgue measure.

For any $M \ge N$, let us consider an arbitrary set $\mathcal{B}^M = \{b_1, \dots, b_M\} \subset \Omega - H^M$ such that the fill-distance $\eta = \eta(M)$ from $\mathcal{B}^M$ to $\Omega - H^M$ verifies

$$\eta = o(M), \ M \to +\infty, \tag{3}$$

and $\mathcal{B}^M$ contains a $\Pi_{m-1}(\mathbb{R}^2)$-unisolvent subset.

Let $V$ be a given non-negative real number.

Now, for $\tau = (\tau_0, \tau_1, \ldots, \tau_m) \in \mathbb{R}^{m+1}$, with $\tau_0, \ldots, \tau_{m-1} \geq 0$ and $\tau_m > 0$, let us consider the functional $\mathcal{J} : H^m(\Omega) \to \mathbb{R}$ defined by

$$J(v) = \langle \rho(v - f) \rangle_M^2 + \tau_0 (\int_{H^M} v(x)dx - V)^2 + \sum_{i=1}^{m} \tau_i |v|_i^2,$$

being $\rho : H^m(\Omega) \to \mathbb{R}^M$ given by $\rho v = (v(b_i))_{i=1,\ldots,M}$.

Observe that the first term of $\mathcal{J}$ measures how well (in the least squares sense) $v$ approximates the values of $f$ over the set $\mathcal{B}^M$, the second term measures how well the volume of $v$ approximates the value $V$ over $H$ while the last term of the sum represent some "minimal energy condition" over the semi-norms $|\cdot|_i$, $i = 1, \ldots, m$, all of them weighted by the parameter vector $\tau$.

Let $\mathcal{H}^N$ the finite-dimensional space generated by the functions

$$\{q_1, \ldots, q_{d(m)}, \Phi_\varepsilon(\cdot\, - a_1), \ldots, \Phi_\varepsilon(\cdot\, - a_N)\}.$$

It verifies that $\mathcal{H}^N$ is a finite dimensional subspace of $H^m(\Omega)$.

**Theorem 1.** *There exists a unique element $\sigma^M \in \mathcal{H}^N$ such that*

$$\mathcal{J}(\sigma) \leq \mathcal{J}(v), \ \forall\, v \in \mathcal{H}^N, \tag{4}$$

*which is also the solution of the following variational problem: Find $\sigma \in \mathcal{H}^N$ such that*

$$\begin{aligned} &\langle \rho\sigma, \rho v \rangle_M + \tau_0 \int_{H^M} \sigma(x)dx \int_{H^M} v(x)dx + \sum_{i=1}^{m} (v, \sigma)_i = \\ &\langle \rho f, \rho v \rangle_M + \tau_0 V \int_{H^M} v(x)dx, \end{aligned} \tag{5}$$

*for all $v \in \mathcal{H}^N$.*

## §3. Computation and convergence

Let $\sigma \in \mathcal{H}^N$ the unique solution of Problem (4). Then $\sigma = \sum_{i=1}^{N+d(m)} \alpha_i \omega_i$, with $\alpha = (\alpha_1, \ldots, \alpha_{N+d(m)}) \in \mathbb{R}^{N+d(m)}$ and

$$\omega_i = \begin{cases} \Phi_\tau(\cdot\, - a_i), & i = 1, \ldots, N, \\ q_{i-N}, & i = N+1, \ldots, N + d(m). \end{cases}$$

By substituting in (6) we find that $\alpha$ is the unique solution of the linear system

$$(\mathbf{A}\mathbf{A}^t + \tau_0 \mathbf{I}_0 \mathbf{I}_0^t) \sum_{i=1}^{m} \tau_i \mathbf{R}_i) \alpha = \mathbf{A}(\rho f)^t + \tau_0 V \mathbf{I}_0^t,$$

where

$$\mathbf{A} = (\rho\omega_i)^t_{i=1,\ldots,N+d(m)}, \quad \mathbf{I}_0 = (\int_{H^M} \omega_i(x)dx)_{1\le i\le N+d(m)},$$
$$\mathbf{R}_k = ((\omega_i,\omega_j)_k)_{1\le i,j\le N+d(m)}, \ k = 1,\ldots,m.$$

**Theorem 2.** *Let us suppose that, in addition to hypotheses (1-3), it holds that*

$$\tau_m = o(M), \ M \to +\infty, \tag{6}$$

$$\tau_i = o(\tau_m), \ M \to +\infty, \ i = 1,\ldots,m-1, \tag{7}$$

$$\frac{Mh^{2m}}{\tau_m} = o(1), \ M \to +\infty \tag{8}$$

*Let $\sigma^M \in \mathcal{H}^N$ the solution of (6) for $V = \int_{H^M} f(x)dx$. Then*

$$\lim_{M\to+\infty} \|f - \sigma^M\|_m = 0.$$

**Remark 3.** Observe that from (6) and (7) we obtain that $h \to 0$ and thus $N \to +\infty$ as $M \to +\infty$.

**Keywords:** approximation, filling holes, splines, variational methods, radial function basis, volume constrains.

**AMS Classification:** 65D07, 65D17, 68U07, 41A15, 41A25, 41A29.

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# On two variable Koornwinder polynomials and three term relations*

Misael Marriaga, Teresa E. Pérez and Miguel A. Piñar

**Abstract**

In 1975, T. Koornwinder introduced a method to generate bivariate orthogonal polynomials by using orthogonal polynomials in one variable. In this work, we study the explicit expressions for the matrix coefficients in their three term relations by using the the three term recurrence relations for the involved univariate orthogonal polynomials. Moreover, some nice examples are considered.

**Keywords:** orthogonal polynomials in two variables, three term relations.
**AMS Classification:** 42C05, 33C50.

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# Asymptotics of the Christoffel functions on the square*

M. Alfaro, A. Peña, M. A. Piñar and M. L. Rezola

**Abstract**

A well-known result in the univariate theory of orthogonal polynomials (see [2]) asserts that if a weight $\omega$ is smooth enough in $(-1,1)$, then the corresponding Christoffel functions satisfy

$$\lim_{n\to\infty} n\,\lambda_n(\omega,x) \;=\; \frac{\omega(x)}{\omega_0(x)}\,, \tag{1}$$

and the convergence is uniform in compact subsets of $(-1,1)$, where $\omega_0$ denotes the Chebyshev weight: $\omega_0(x) \;=\; \frac{1}{\pi\sqrt{1-x^2}}\,,\; x\in(-1,1)\,.$

Having in mind this result, it is natural to expect that for sufficiently "regular" weights in the bivariate case, it holds

$$\lim_{n\to\infty}\binom{n+2}{n}\Lambda_n(W;x,y) = \frac{W_(x,y)}{W_0(x,y)}, \tag{2}$$

uniformly in compact subsets of $\Omega$, where $W_0$ is an analogue of the Chebyshev weight for this domain. However ssymptotics for the multivariate Christoffel functions have been established just in a very few cases (see [1])

- In the unit ball in $\mathbb{R}^d$, for Jacobi weights, and weights that satisfy some structural restriction, such as being radially or centrally symmetric.
- In the standard simplex in $\mathbb{R}^d$ for Jacobi weights.

We will establish (2) for weights on the square $S$. Here, the analogous of Chebyshev weight is given by

$$W_0(x,y) \;=\; \frac{1}{\pi}\,(1-x^2)^{-1/2},(1-y^2)^{-1/2}\,,\; -1<x<1\,,\; -1<y<1\,.$$

which is normalized in such a way that $\int_S W_0(x,y)\,dx\,dy \;=\; 1\,.$

**Keywords:** orthogonal polynomials in several variables, Christoffel functions.
**AMS Classification:** 42C05, 33C50.

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# Can we cut down Runge oscillations?

Christophe Rabut

**Abstract**

Given $n+1$ data points, we show in this talk various polynomials interpolating these data which oscillate less, or far less than the usual (unique) degree $n$ polynomial interpolating the data. At this stage of the work on this subject, this is only experimental trials which however seem to validate the various ideas presented to cut down oscillations, still using true polynomial interpolation. So we do not have any convergence result (even if we have some numerical evidence about that). Just to keep on with some kind of suspense, I prefer not expliciting them here. I intend to ask the audience if they know or not the methods (probably three methods) I will present during the talk.

For a better computation stability, all computations are done in the Bernstein basis.

**Keywords:** Runge phenomenon, polynomial interpolation, polynomials with variational properties.

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# Sharp Szegö-type coefficient estimates for multivariate polynomials

Heinz-Joachim Rack

**Abstract**

Let $P_n$ denote a real univariate polynomial of degree $\leq n$ with $P_n(x) = \sum_{k=0}^{n} a_{n,k}x^k, n \geq 1$, and let $T_n$ with $T_n(x) = \sum_{k=0}^{n} t_{n,k}x^k$ and explicitly known $t_{n,k}$'s denote the $n$-th Chebyshev polynomial of the first kind [23]. V. A. Markov in his celebrated paper of 1892 [7, pp. 80 - 81] has discovered a two-staged extremal coefficient property of $T_n$ resp. of $T_{n-1}$ when compared to an arbitrary $P_n$ satisfying $||P_n||_\infty \leq 1$ on the interval $[-1; 1]$, where $|| \cdot ||_\infty$ denotes the uniform norm, see also [9, p. 384], [22, pp. 672 - 673], [23, p. 147]:

$$|a_{n,k}| \leq |t_{n,k}|, \text{ if } n-k \text{ is even,} \tag{1}$$

$$|a_{n,k}| \leq |t_{n-1,k}|, \text{ if } n-k \text{ is odd} \tag{2}$$

(here (2) is a corollary of (1)). The problem to find the sharp upper bound for $|a_{2,k}|, k \in \{0, 1, 2\}$, had been explicitly posed by the renowned chemist Mendeleev in 1887 [8, p. 289], [16]. The famous case $k = n$ (i.e. $|a_{n,n}| \leq |t_{n,n}| = t_{n,n} = 2^{n-1}$) traces back to Chebyshev's pioneering paper of 1854 [2, p. 123], see also [9, p. 385; p. 423], [20], [23, p. 68]. An inequality of Szegö for pairs of consecutive coefficients of $P_n$ (with $||P_n||_\infty \leq 1$), as communicated in 1947 by Erdös [3, p. 1176], see also [16] and [22, Theorem 16.3.3], ingeniously extends (1):

$$|a_{n,k-1}| + |a_{n,k}| \leq |t_{n,k}|, \text{ if } n-k \text{ is even.} \tag{3}$$

We have in turn found extensions of (3) to complementary pairs $|a_{n,k}|+|a_{n,k+1}|$ of consecutive coefficients of $P_n$, and more generally, to pairs of coefficients of $P_n$ (with $||P_n||_\infty \leq 1$) which need not be adjacent to each other:

$$|a_{n,k}| + |a_{n,j}| \leq |t_{n,k}|, \text{ if } n-k \text{ is even and } n-j \text{ is odd with } k+1 \leq j, \tag{4}$$

$$|a_{n,j}| + |a_{n,k}| \leq |t_{n,k}|, \text{ if } n-k \text{ is even and } n-j \text{ is odd with } j \leq k-3, \tag{5}$$

see [19, 21] for accessory assumptions on $k$ and $n$. In [11] - [18] we have considered generalizations of (1), (2), (3) to multivariate polynomials, see also [1, 4, 5, 6, 10, 24]. In the present talk we will provide generalizations of (4), (5) to multivariate polynomials.

**Keywords:** Chebyshev, coefficient, estimate, extremal, homogeneous, inequality, Markov, Mendeleev, multivariate, pair of coefficients, polynomial, problem, several variables, Szegö, Szegö-type.

**AMS Classification:** 26D05, 41A10, 41A63.

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# Optimal cubic Lagrange interpolation: extremal node systems in [-1, 1] with minimal Lebesgue constant

Heinz-Joachim Rack and Robert Vajda

**Abstract**

A two-fold solution to the cubic case of optimal polynomial Lagrange interpolation is given. In particular, two explicit analytical descriptions are provided for the uncountable infinitely many extremal 4-point node configurations in $[-1, 1]$ which all lead to the (known) minimal Lebesgue constant of cubic Lagrange interpolation. The proofs are guided by symbolic computation. In closing, the quadratic and the quartic case will be briefly touched upon.

**Keywords:** constant, cubic, extremal, interpolation, Lagrange interpolation, Lebesgue constant, minimal, node, node system, optimal, point, polynomial, symbolic computation.
**AMS Classification:** 05C35, 33F10, 41A05, 41A44, 65D05, 68W30.

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# Interpolation on several intervals*

József Szabados

**Abstract**

Approximation theoretic problems related to a set of finitely many intervals on the real line usually constitute unexpected difficulties. Markov and Bernstein type inequalities for derivatives of polynomials have been thoroughly investigated. In this talk we initiate a generalization of linear operators approximating continuous functions. In particular, we consider the order of magnitude of the Lebesgue constant of Lagrange interpolation, it being the deciding factor of convergence of the interpolation process. It turns out that, apart from the simplest case of two intervals of equal lengths, the problem of constructing good systems of points of interpolation is difficult. The main tool in this respect will be the so-called T-polynomials introduced by Franz Peherstorfer. Hermite–Fejér interpolation on two intervals of equal length will also be considered.

Most of the talk is based on joint works with Alexey Lukashov and András Kroó.

**Keywords:** Lagrange interpolation, Lebesgue constant, T-polynomial, Hermite–Fejér interpolation.

**AMS Classification:** 41A05, 41A35.

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# Progress on the Gasca-Maeztu Conjecture

Hakop Hakopian, Kurt Jetter and Georg Zimmermann

**Abstract**

An $n$-poised set in two dimensions is a set of nodes admitting unique bivariate interpolation with polynomials of total degree at most $n$. We are interested in poised sets with the property that all fundamental polynomials are products of linear factors. In 1982, M. Gasca and J. I. Maeztu [1] conjectured that every such set necessarily contains $n+1$ collinear points. The case $n = 4$ was proved for the first time in 1990 by J. R. Busch [2], later with different methods by J. M. Carnicer and M. Gasca [3], and later again with different methods by the authors [4]. The case $n = 5$ was shown by the authors in [5]. We present the latest progress in the attempt to prove the general result.

**Keywords:** polynomial interpolation, Gasca-Maeztu conjecture, fundamental polynomial, maximal line, poised set.
**AMS Classification:** 41A05, 41A63, 14H50.

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