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NONLINEAR n-TERM APPROXIMATION ON THE SPHERE FROM SHIFTS OF THE NEWTONIAN KERNEL

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Abstract

The fundamental solution of the Laplace equation (Newtonian kernel) $\frac{1}{|x|^{d-2}}$ in dimension $d > 2$ or $\ln \frac{1}{|x|}$ if $d = 2$ is a basic building block in Potential theory. The main goal of this lecture is to study the rates of nonlinear n-term approximation of $\mathcal{H}^p, 0 < p < \infty$, and BMO functions on the unit sphere $\mathbb{S}^{d-1}$ from shifts of the Newtonian kernel with poles outside the unit ball $\overline{B^d} \subset \mathbb{R}^d$. We prove the following Jackson-type estimates.

Theorem 1. *Let $0 < p < \infty$, $s > 0$, $1/\tau = s/(d-1) + 1/p$. If $f \in \mathcal{B}^{s\tau}_\tau(\mathbb{S}^{d-1})$, then $f \in \mathcal{H}^p(\mathbb{S}^{d-1})$ and for $n \geq 1$*

$$E_n(f)_{\mathcal{H}^p} \leq c n^{-s/(d-1)} \|f\|_{\mathcal{B}^{s\tau}_\tau},$$

where the constant $c > 0$ depends only on p, s, d.

Theorem 2. *Let $s > 0$, $1/\tau = s/(d-1)$. If $f \in \mathcal{B}^{s\tau}_\tau(\mathbb{S}^{d-1})$, then $f \in \mathrm{BMO}(\mathbb{S}^{d-1})$ and for $n \geq 1$*

$$E_n(f)_{\mathrm{BMO}} \leq c n^{-s/(d-1)} \|f\|_{\mathcal{B}^{s\tau}_\tau},$$

where the constant $c > 0$ depends only on s, d.

Here $\mathcal{H}^p(\mathbb{S}^{d-1}), 0 < p < \infty$, denotes the Hardy space on $\mathbb{S}^{d-1}$ ($\mathcal{H}^p = L^p$ for $1 < p < \infty$), $\mathrm{BMO}(\mathbb{S}^{d-1})$ denotes the Bounded mean oscillation space on $\mathbb{S}^{d-1}$ and $\mathcal{B}^{sq}_p(\mathbb{S}^{d-1})$ stands for the Besov space with parameters s, q, p on the sphere. $E_n(f)_{\mathfrak{B}}$ denotes the best nonlinear n-term approximation of f from shifts of the Newtonian kernel in the norm of the Banach space $\mathfrak{B}$.

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The rates of approximation in Theorems 1 and 2 are optimal in terms of the Besov spaces. The main vehicle in establishing these results is the construction of highly localized frames for Besov and Triebel-Lizorkin spaces on the sphere whose elements are linear combinations of a fixed number of shifts of the Newtonian kernel.

The above results naturally extends to approximation of harmonic functions in B^d. According to Theorem 2 BMO is a natural replacement for L^∞ in best approximation relations.

Keywords: Nonlinear approximation, harmonic functions, Newtonian kernel, Hardy spaces, Besov spaces, frame decomposition.

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