HOLE FILLING SPLINES WITH VOLUMEN CONSTRAINS BY RADIAL BASIS FUNCTIONS

M. PASADAS, M.A. FORTES, P. GONZÁLEZ AND A. PALOMARES

Abstract

In many situations we have to fill one or several holes of certain function defined in a domain where there is a lack of information inside some sub-domains. For this we have developed some methods (see [1, 2, 3, 4]).

But in some practical cases we just know some specific geometrical constraints, of industrial or design type, as the special case of a specified volume inside each one of these sub-domains. In this work we study this particular issue, giving both some theoretical and computational results that assures the feasibility of the corresponding procedures.

The studied method in this work manage to find a function of a vector space generated by a radial function basis that minimizes certain quadratic functional that includes some terms associated with the volume constrain and the usual semi-norms in a Sobolev space. In this way, some approximation methods have been developed (see [5, 6]).

In next Section 2 we establish some general and specific notation as the functional spaces where we obtain the reconstructed functions. In section 3 we pose the problem of finding a function that fill a given hole and fulfills a volume restriction. In Section 4 we establish the computation algorithm and a convergence result.

Keywords: Approximation, filling holes, splines, variational methods, radial function basis, volume constrains.


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1. Preliminaries and notation

Let $m \geq 1$ be a positive integer and let $\Pi_{m-1}(\mathbb{R}^2)$ denote the space of polynomials on $\mathbb{R}^2$ of degree at most $m-1$ whose dimension is $d(m) = \frac{m(m+1)}{2}$. Let $\{q_1, \ldots, q_{d(m)}\}$ the standard basis of $\Pi_{m-1}(\mathbb{R}^2)$.

Consider the following function
\[ \phi_\varepsilon(t) = -\frac{1}{2\varepsilon^3} \left( e^{-\varepsilon \sqrt{t}} + \varepsilon \sqrt{t} \right), \quad \varepsilon \in \mathbb{R}^+ \]
and the following radial function
\[ \Phi_\varepsilon(x) = \phi_\varepsilon(\langle x \rangle_2^2) = -\frac{1}{2\varepsilon^3} \left( e^{-\varepsilon \langle x \rangle_2^2} + \varepsilon \langle x \rangle_2 \right), \quad x \in \mathbb{R}^2, \]
where $\langle \cdot \rangle_k$ is the Euclidean norm in $\mathbb{R}^k$.

Let $\Omega$ be an open bounded nonempty subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary.

We will use the classical notation $H^k(\Omega)$ to denote the usual Sobolev space of all distributions $u$ which all of whose derivatives up to and including order $k$ are in the classical Lebesgue space $L^2(\Omega)$.

The Sobolev space $H^k(\Omega)$ is a Hilbert space equipped with the inner semi-products given by
\[ (u, v)_\ell = \sum_{|\alpha| = \ell} \int_\Omega D^\alpha u(x) D^\alpha v(x) d, \quad 0 \leq \ell \leq k, \]
the semi-norms given by $|u|_\ell = (u, u)_\ell$, for all $\ell = 0, \ldots, k$, and the norm $\|u\|_k = \left( \sum_{\ell \leq k} |u|_\ell^2 \right)^{\frac{1}{2}}$.

2. Hole filling meshfree smoothing spline surface with volume constraint

For any $N \geq d(m)$, let us an arbitrary set $\mathcal{A}^N = \{ a_1, \ldots, a_N \} \subset \mathbb{R}^2$ such that it contains a $\Pi_{m-1}(\mathbb{R}^2)$-unisolvent subset (i.e., if $q \in \Pi_{m-1}(\mathbb{R}^2)$ and for all $a \in \mathcal{A}^N$, $q(a) = 0$ then $q = 0$).

Let $h = h(N)$ be the fill-distance from $\mathcal{A}^N$ to $\Omega$ defined by
\[ h = \sup_{x \in \Omega} \inf_{a \in \mathcal{A}^N} \langle x - a \rangle_2 \]
and suppose that
\[ h = o(N), \quad N \to +\infty. \]
On the other hand, for any $M \geq N$ let $H^M$ (the hole) be an open nonempty subset of $\Omega$ verifying

\[ \mu(H^M) = o(1), \quad M \to +\infty, \]

where $\mu$ represents the Lebesgue measure.

For any $M \geq N$, let us consider an arbitrary set $B^M = \{b_1, \ldots, b_M\} \subset \Omega - H^M$ such that the fill-distance $\eta = \eta(M)$ from $B^M$ to $\Omega - H^M$ verifies

\[ \eta = o(M), \quad M \to +\infty, \]

and $B^M$ contains a $\Pi_{m-1}(\mathbb{R}^2)$-unisolvent subset.

Let $V$ be a given non-negative real number.

Now, for $\tau = (\tau_0, \tau_1, \ldots, \tau_m) \in \mathbb{R}^{m+1}$, with $\tau_0, \ldots, \tau_{m-1} \geq 0$ and $\tau_m > 0$, let us consider the functional $J : H^m(\Omega) \to \mathbb{R}$ defined by

\[ J(v) = \langle \rho(v - f) \rangle_M^2 + \tau_0 \int_{H^M} v(x)dx - V)^2 + \sum_{i=1}^m \tau_i |v_i|^2, \]

being $\rho : H^m(\Omega) \to \mathbb{R}^M$ given by $\rho v = (v(b_i))_{i=1,\ldots,M}$.

Observe that the first term of $J$ measures how well (in the least squares sense) $v$ approximates the values of $f$ over the set $B^M$, the second term measures how well the volume of $v$ approximates the value $V$ over $H$ while the last term of the sum represents some “minimal energy condition” over the semi-norms $|\cdot|_i$, $i = 1, \ldots, m$, all of them weighted by the parameter vector $\tau$.

Let $H^N$ the finite-dimensional space generated by the functions

\[ \{q_1, \ldots, q_{d(m)}, \Phi(\cdot - a_1), \ldots, \Phi(\cdot - a_N)\}. \]

It verifies that $H^N$ is a finite dimensional subspace of $H^m(\Omega)$.

**Theorem 1.** There exists a unique element $\sigma^M \in H^N$ such that

\[ J(\sigma) \leq J(v), \quad \forall \, v \in H^N, \]

which is also the solution of the following variational problem: Find $\sigma \in H^N$ such that

\[ \langle \rho\sigma, \rho v \rangle_M + \tau_0 \int_{H^M} \sigma(x)dx \int_{H^M} v(x)dx + \sum_{i=1}^m (v, \sigma)_i = \]

\[ \langle \rho f, \rho v \rangle_M + \tau_0 V \int_{H^M} v(x)dx, \]

for all $v \in H^N$.

Observe
3. Computation and convergence

Let \( \sigma \in H^N \) the unique solution of Problem (4). Then \( \sigma = \sum_{i=1}^{N+d(m)} \alpha_i \omega_i \), with \( \alpha = (\alpha_1, \ldots, \alpha_{N+d(m)}) \in \mathbb{R}^{N+d(m)} \) and

\[
\omega_i = \begin{cases} 
\Phi_r(\cdot - a_i), & i = 1, \ldots, N, \\
q_{i-N}, & i = N + 1, \ldots, N + d(m).
\end{cases}
\]

By substituting in (6) we find that \( \alpha \) is the unique solution of the linear system

\[
(AA^t + \tau_0 I_0 I_0^t) \sum_{i=1}^m \tau_i R_i) \alpha = A(\rho f)^t + \tau_0 V I_0^t,
\]

where

\[
A = (\rho \omega_i)_{i=1,\ldots,N+d(m)}, \quad I_0 = (\int_{H^M} \omega_i(x)dx)_{1 \leq i \leq N+d(m)}, \quad R_k = ((\omega_i, \omega_j)_k)_{1 \leq i, j \leq N+d(m)}, \quad k = 1, \ldots, m.
\]

Theorem 2. Let us suppose that, in addition to hypotheses (1-3), it holds that

\[
\tau_m = o(M), \quad M \to +\infty,
\]

(6)

\[
\tau_i = o(\tau_m), \quad M \to +\infty, \quad i = 1, \ldots, m-1,
\]

(7)

\[
\frac{Mh^{2m}}{\tau_m} = o(1), \quad M \to +\infty.
\]

(8)

Let \( \sigma^M \in H^N \) the solution of (6) for \( V = \int_{H^M} f(x)dx \). Then

\[
\lim_{M \to +\infty} \| f - \sigma^M \|_m = 0.
\]

Remark 1. Observe that from (6) and (7) we obtain that \( h \to 0 \) and thus \( N \to +\infty \) as \( M \to +\infty \).

Bibliography


M. Pasadas,
Departamento de Matemática Aplicada, ETSI Caminos, C. y P.,
Universidad de Granada, 18071 Granada.
pasadas@ugr.es

M.A. Fortes,
Departamento de Matemática Aplicada, ETSI Caminos, C. y P.,
Universidad de Granada, 18071 Granada.
fortes@ugr.es

P. González,
Departamento de Matemática Aplicada, ETSI Caminos, C. y P.,
Universidad de Granada, 18071 Granada.
prodelas@ugr.es

A. Palomares,
Departamento de Matemática Aplicada, ETSI Caminos, C. y P.,
Universidad de Granada, 18071 Granada.
apalom@ugr.es