On Bernstein and Chebyshev type problems for \( k \)-monotone polynomials\(^\dagger\)

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Abstract

In a recent paper [4], we initiated the study of Markov type inequalities for the so called \( k \)-monotone polynomials of degree at most \( n \), whose first \( k \) derivatives are nonnegative in the interval considered. The exact constant of Markov type inequality was found in case of first derivative when the underlying norm is uniform or \( L_1 \). Moreover it was shown that in general for derivatives of order \( j \) the sharp order of magnitude of Markov constants is \( \left( \frac{n^2}{k} \right)^j \).

In the present paper we continue the study of this subject by establishing asymptotically sharp Bernstein type inequalities for derivatives of \( k \)-monotone polynomials. We shall show that the order of magnitude of Bernstein factors is \( \left( \varphi \left( \frac{k}{n^2} \right) \frac{n^2}{k} \right)^j \), with the size of the derivatives on \([-1, 1]\) being measured in weighted norms with weight \( \varphi(1 - x) \), where \( \varphi \) is such that \( \varphi(x) \) is increasing and \( \varphi(x)/x \) is decreasing.

In addition we also give an exact solution to the Chebyshev type problem of finding monic \( k \)-monotone polynomials of minimal \( L_1 \) norm. It turns out that just as in the case of Markov type inequalities for \( k \)-monotone polynomials, these extremal polynomials are related to certain Jacobi polynomials. The result for \( L_\infty \) norm of Bernstein will also be recovered.

Keywords: Bernstein inequality, Chebyshev problem, \( k \)-convex polynomial.

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