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Ultraspherical and pseudo-ultraspherical polynomials

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Abstract

The pseudo-ultraspherical polynomial of degree n is defined by $\tilde{C}_n^{(\lambda)}(x) = (-i)^n C_n^{(\lambda)}(ix)$ where $C_n^{(\lambda)}(x)$ is the ultraspherical polynomial. We discuss the orthogonality of finite sequences of pseudo-ultraspherical polynomials $\{\tilde{C}_n^{(\lambda)}(x)\}_{n=0}^{N}$ for different values of N that depend on λ. We use an identity linking the zeros of $\tilde{C}_n^{(\lambda)}(x)$ and the zeros of $C_n^{(\lambda')}(x)$ where $\lambda' = 1/2 - \lambda - n$ to prove that for $1 - n < \lambda < 2 - n$ and each $n \geq 3$, there are two zeros of $\tilde{C}_n^{(\lambda)}(x)$ which are purely imaginary and symmetric with respect to the origin. We derive upper and lower bounds for the distance of the two purely imaginary zeros of $\tilde{C}_n^{(\lambda)}(x)$ from the origin when $1 - n < \lambda < 2 - n$.

Keywords: orthogonality, finite orthogonal sequences, pseudo-ultraspherical polynomials, interlacing zeros, purely imaginary zeros, Wendroff's Theorem, Favard's Theorem.

MSC: Primary 33C50; Secondary 41A25, 42C05.

§1. Introduction

The sequence of monic ultraspherical (also called Gegenbauer) polynomials $\{C_n^{(\lambda)}(x)\}_{n=0}^{\infty}$ can be defined by the three term recurrence relation [10, (8.18)]

$$C_n^{(\lambda)}(x) = xC_{n-1}^{(\lambda)}(x) - \frac{(n-1)(n+2\lambda-2)}{4(n+\lambda-2)(n+\lambda-1)}C_{n-2}^{(\lambda)}(x), \tag{1.1}$$