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Inequalities for the exponential function

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Abstract

We study the log-concavity of the function $a \mapsto a^\delta J_a(x)$, where

$$J_a(x) = e^x - \left(1 + \frac{x}{a}\right)^a$$

and use our result to show that $J_a(x)$ satisfies certain functional inequalities. Among others, we prove that if a, b and λ, μ are positive real numbers with $a \neq b$ and $\lambda + \mu = 1$, then we have for all $x > 0$,

$$\frac{a^\lambda b^\mu}{\lambda a + \mu b} < \frac{J_{\lambda a + \mu b}(x)}{J_a(x)^\lambda \, J_b(x)^\mu}.$$

The lower bound is best possible.

Keywords: exponential function, inequalities, concave, convex, sub- and superadditive.

MSC: 26D07, 39B62.

§1. Introduction

The classical exponential function

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$